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(Second) Quantised resolvents and regularised traces

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Abstract

Regularised traces on classical pseudodifferential operators are extended to tensor products of classical pseudodifferential operators via a *(second) quantisation procedure*. Whereas ordinary ζ-regularised traces are not generally expected to be local, using techniques borrowed from Connes and Moscovici [A. Connes, H. Moscovici, The local index formula in noncommutative geometry, Geom. Funct. Anal. 5 (2) (1995) 174–243], Higson [N. Higson, The residue index theorem of Connes and Moscovici, in: Clay Mathematics Proceedings, 2004, http://www.math.psu.edu/higson/ResearchPapers.html], we show that if Q has scalar leading symbol, higher *quantised ζ-regularised traces* are local since they can be expressed as a finite linear combination of noncommutative residues. Just as ordinary ζ-regularised traces, they present anomalies (Hochschild coboundary, dependence on the weight Q), which for quantised ζ-regularised traces of level n, are roughly speaking finite linear combinations of quantised regularised traces of level $n+1$. As a result, anomalies are local for any non negative n, which yields back as a particular case the fact that ordinary ζ-regularised traces present local anomalies.[1]

Keywords: Traces; Pseudodifferential operators; Regularisation; Noncommutative residue

0. Introduction

Ordinary ζ-regularised traces have been the object of many investigations (see e.g. the works of Grubb and Seeley [12], Kontsevich and Vishik [17], Melrose and Nistor [20], Lesch [18], and more recent works by Grubb [9–11], as well as recent papers by Scott and the author [26,27]). Regularised traces naturally arise in the study of variations of partition functions in quantum field theory and provide useful tools in the context of anomalies (see e.g. [4,19,5]). They also occur in the framework of index theory, specifically in the local index formula of Connes and Moscovici (from whom we borrow some of the techniques used in this paper) in noncommutative geometry [7], in the fractional index theory of Mathai, Melrose and Singer [21] as well as in the family index theorem see e.g. [29,26,22].

Whereas regularised traces are not expected to be local [27], their variations are. Viewing regularised traces as quantised regularised traces, namely as higher order cochains on the algebra of classical pseudo-differential operators, sheds light on this fact, combining two observations:

1. variations of regularised traces of level n are regularised traces of level $n+1$,

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[1] This work is partially based on [S. Paycha, Weighted trace cochains; A geometric setup for anomalies, Max Planck Institute, 2005. Preprint] although we use other conventions here which lead to slightly different definitions.

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2. quantised traces of positive level (i.e. positive order cochains) are local.

The well-known locality property of anomalies for ordinary ζ-regularised traces [17,20,6,5,23] then arises as a consequence of the locality of quantised traces of positive level.

Whereas ζ-regularisation and heat-kernel regularisation lead to the same regularised traces only on operators with vanishing noncommutative residue, higher quantised ζ-regularised traces coincide with higher quantised heat-kernel regularised traces on the whole algebra of classical pseudodifferential operators for high enough quantum level n. Heat-kernel regularised traces naturally arise from JLO type cochains.[2] Their analogues in the noncommutative context arise in the work of Connes and Moscovici [7] (later reformulated by Higson [14]) on the local formula for the Connes–Chern character.

Let us briefly describe the "second quantisation" procedure for ζ-regularised traces. Let $\mathcal{C}_\bullet(M,E) := \oplus_{n=0}^{\infty}\mathcal{C}_n(M,E)$ with $\mathcal{C}_n(M,E) = \otimes^{n+1} C\ell(M,E)$ be the space of chains associated to the algebra $C\ell(M,E)$ of classical pseudodifferential operators acting on smooth sections of a vector bundle E over a closed manifold M.

The resolvent $R(\lambda, Q) = (\lambda - Q)^{-1}$ of an operator $Q \in C\ell(M,E)$ can be quantised to $R_\bullet(\lambda, Q+\theta)$ acting on $\mathcal{C}_\bullet(M,E)$, θ being an insertion map $\theta(A) = A$. We set $R_0(\lambda, Q+\theta) = R(\lambda, Q)$ and for $n > 0$

$$\begin{aligned} R_n(\lambda, Q+\theta):\quad \mathcal{C}_{n-1}(M,E) &\to C\ell(M,E)\\ A_1\otimes\cdots\otimes A_n &\mapsto R(\lambda,Q)A_1\cdots R(\lambda,Q)A_n\,R(\lambda,Q). \end{aligned}$$

In Section 2 we construct quantised functionals $f(Q+\theta)$ using Cauchy integrals (see Theorem 1)

$$f_\bullet(Q+\theta) = \frac{1}{2i\pi}\int f(\lambda)\,R_\bullet(\lambda, Q+\theta)$$

along an adequately chosen contour. In Section 3, we investigate the behaviour of $R_\bullet(\lambda, Q+\theta)$ under the adjoint action $A \mapsto [C, A]$ of $C\ell(M,E)$ and under a variation of the weight, from which we derive the corresponding behaviour of the quantised functionals $f(Q+\theta)$.

Quantised functionals are the cornerstones for the quantisation of ζ-regularised traces.

Let us first recall the usual ζ-regularisation procedure. If Q is elliptic with spectral cut, it has well defined complex powers defined by Cauchy integrals. Given an operator $A \in C\ell(M,E)$ and a complex number z with large enough real part, the ζ-regularised operator $R^{Q,\zeta}(A)(z) := A\,Q^{-z}$ is trace class (we assume Q is invertible for simplicity) and the map $z \mapsto \mathrm{tr}(A\,Q^{-z})$ extends to a meromorphic function $z \mapsto \zeta(A,Q,z)\,|^{\mathrm{mer}}$ with simple pole at $z=0$ (see e.g. [17]). Its finite part at $z=0$ gives rise to a linear map $\mathrm{tr}^Q : C\ell(M,E) \to \mathbb{C}$ which we call the Q-weighted trace. In general, the finite part $\mathrm{tr}^Q(A)$ is not expected to be local[3] in contrast to the complex residue at $z=0$, which is proportional to the noncommutative residue [33].

Replacing the resolvent $R(\lambda,Q)$ by the quantised resolvent $R_\bullet(\lambda, Q+\theta)$ boils down to substituting the quantised complex powers $(Q+\theta)^{-z}$ to ordinary complex powers Q^{-z} and leads to a *quantised zeta regularisation* $\mathcal{R}^{Q+\theta,\zeta}$ on $\mathcal{C}_\bullet(M,E)$ defined in terms of the quantised resolvent by

$$R_n^{Q+\theta,\zeta}(A_0\otimes A_1\otimes\cdots\otimes A_n)(z) = \frac{1}{2i\pi}\int_\Gamma \lambda^{-z}A_0R_n(\lambda, Q+\theta)(A_1\otimes\cdots\otimes A_n). \tag{1}$$

Using techniques inspired from [14,7], when Q has *scalar leading symbol*, we show that for any $A_i \in C\ell(M,E)$, $i = 0,\ldots,n$ the operator $\mathcal{R}_n^{Q+\theta,\zeta}(A_0\otimes A_1\otimes\cdots\otimes A_n)(z)$ is trace class for z with large enough real part. Furthermore, the map $z \mapsto \mathrm{tr}(\mathcal{R}_n^{Q,\zeta}(A_0\otimes A_1\otimes\cdots\otimes A_n))(z)$ extends to a meromorphic function with simple pole at $z=0$. It is holomorphic at $z=0$ when $n \neq 0$ (see Theorem 2) and we call the (second) *quantised weighted ζ-trace (or quantised Q-weighted trace)* of the chain $A_0\otimes\cdots\otimes A_n$ its value at $z=0$ which we denote by $\mathrm{tr}_n^{Q+\theta}(A_0\otimes\cdots\otimes A_n)$. In contrast to the ordinary Q-weighted trace, we show that whenever Q has scalar leading symbol, quantised weighted traces $\mathrm{tr}_n^{Q+\theta}$ are local for any positive integer n. We provide a *local formula* expressing them as a finite linear combination of noncommutative residues (see Theorem 3). When transposed to the noncommutative context, the locality for positive

[2] Named after Jaffe, Lesniewski and Osterwalder [16].

[3] It actually is local if A is a differential operator. In general, it is made of a local piece involving the noncommutative residue and a global piece involving a finite part integral over all the cotangent space [27].

integer n shown here underlies that of the Connes–Moscovici formula for the Connes–Chern character in the case of classical pseudodifferential operators.

When Q has positive leading symbol, if instead of ζ-regularisation we implement heat-kernel regularisation (thus replacing Q^{-z} by $\mathrm{e}^{-\varepsilon Q}$ for some positive parameter ε), a similar construction gives rise to JLO type cochains. The heat-kernel quantised traces one obtains this way (taking finite parts as ε tends to 0) coincide for large quantum level with the quantised weighted trace described previously (see Theorem 3). This again constrasts with the 0 level case. Indeed, the finite part of the heat-kernel regularised trace of an operator $A \in C\ell(M, E)$ only coincides with its weighted trace when the operator A has vanishing residue.

Quantised Q-weighted traces are not generally closed in the Hochschild cohomology. When Q has *scalar leading symbol*, we show that their *Hochschild coboundary* is local as a finite linear combination of noncommutative residues (see Theorem 5). For even cochains, this follows from the fact that the Hochschild coboundary of a quantised regularised trace of level $2p$ is a linear combination of quantised regularised traces of level $2p+1$ (see Proposition 7).[4]

We also express the variation of such quantised weighted traces as the *weight varies* in terms of a linear combination of noncommutative residues. This locality is again a consequence of the fact that the variation of a quantised weighted trace of weight n is a linear combination of weighted traces of weight $n + 1$ (see Theorem 6) combined with the locality of quantised regularised traces of any positive level.

Adapting these constructions to the geometric setup of the index theorem for *families* along the lines of [26,22], Q can be replaced by a pseudodifferential operator-valued even form $\mathbf{Q}$, the exterior diffferentiation by a superconnexion $\mathbb{A}$ and the pseudodifferential operators A_i by pseudodifferential operator-valued forms α_i (the insertion map θ for pseudodifferential operators is replaced by an insertion map Θ for pseudodifferential valued forms) and one gets a local expression for the exterior differential $(\mathrm{d}\,\mathrm{tr}_n^{\mathbf{Q}+\Theta})$ of quantised regularised traces (see Theorem 7).

When $\mathbb{A}$ is a superconnection adapted to the zero degree component $Q = \mathbf{Q}_{[0]}$, replacing $\mathbf{Q}$ by $\mathbb{A}^2$ in the above expression yields the expected covariance property for quantised $\mathbb{A}^2$-weighted traces (see Corollary 11).

To sum up, local formulae for the two types of anomalies mentioned above, lack of traciality and dependence on the weight of the quantised regularised traces, are obtained in the same manner, namely as a combination of the following basic facts:

1. *Anomalies for quantised regularised traces of level n are linear combinations of quantised regularised traces of level $n + 1$.*[5]
2. *Quantised regularised traces of any positive level are local.*

As mentioned above, quantised regularised traces are tools commonly used in noncommutative geometry; some of the above constructions inspired from techniques used in noncommutative geometry generalise to a noncommutative context. However we feel that even in the present classical setup of classical pseudodifferential operators, the language of quantised regularised traces is well suited to keeping track of anomalies. The locality of quantised regularised traces (of any positive level) shown in this paper somewhat clarifies why one is to expect trace anomalies to be local.

1. Prerequisites on classical pseudodifferential symbols and operators

We briefly recall some notions concerning symbols and pseudodifferential operators and fix the corresponding notations. Classical references for the polyhomogeneous symbol calculus are e.g. [8,12,15,31,32].

1.1. Classical symbols and operators

In the sequel, E denotes a smooth Hermitian vector bundle based on some closed Riemannian manifold M. The space $C^\infty(M, E)$ of smooth sections of E is endowed with the inner product $\langle \psi, \phi \rangle := \int_M \mathrm{d}\mu(x) \langle \psi(x), \phi(x) \rangle_x$ induced by the Hermitian structure $\langle \cdot, \cdot \rangle_x$ on the fibre over $x \in M$ and the Riemannian measure μ on M.

Given an open subset U of $\mathbb{R}^n$ and an auxiliary (finite-dimensional) normed vector space V, the set of symbols $\mathrm{S}^r(U, V)$ on U of order $r \in \mathbb{R}$ consists of those functions $\sigma(x, \xi)$ in $C^\infty(T^*U, \mathrm{End}(V))$ such that $\partial_x^\mu \partial_\xi^\nu \sigma(x, \xi)$

[4] This fact holds only in the even case $n = 2p$, but the locality still holds in the odd case $n = 2p + 1$.

[5] Here we have left aside the Hochschild coboundary on odd cochains (see the previous footnote).

is $O((1+|\xi|)^{r-|\nu|})$ for all multi-indices μ, ν, uniformly in ξ, and, on compact subsets of U, uniformly in x. We set $\mathrm{S}(U,V) := \bigcup_{r\in\mathbb{R}} \mathrm{S}^r(U,V)$ and $\mathrm{S}^{-\infty}(U,V) := \bigcap_{r\in\mathbb{R}} \mathrm{S}^r(U,V)$. A *classical* (polyhomogeneous) symbol of order $\alpha \in \mathbb{C}$ means a function $\sigma(x,\xi)$ in $C^\infty(T^*U, \mathrm{End}(V))$ such that for each $N \in \mathbb{N}$ and each integer $0 \leq j \leq N$ there exists $\sigma_{\alpha-j} \in C^\infty(T^*U, \mathrm{End}(V))$ which is homogeneous in ξ of degree $\alpha - j$ for $|\xi| \geq 1$, so $\sigma_{\alpha-j}(x, t\xi) = t^{\alpha-j}\, \sigma_{\alpha-j}(x,\xi)$ for $t \geq 1$, $|\xi| \geq 1$, and a symbol $\sigma_{(N)} \in \mathrm{S}^{\mathrm{Re}(\alpha)-N-1}(U,V)$ such that

$$\sigma(x,\xi) = \sum_{j=0}^{N} \sigma_{\alpha-j}(x,\xi) + \sigma_{(N)}(x,\xi) \quad \forall (x,\xi) \in T^*U. \tag{2}$$

We then write $\sigma(x,\xi) \sim \sum_{j=0}^{\infty} \sigma_{\alpha-j}(x,\xi)$. Let $\mathrm{CS}(U,V)$ denote the class of classical symbols on U with values in V and let $\mathrm{CS}^\alpha(U,V)$ denote the subset of classical symbols of order α.

When $V = \mathbb{C}$, we write $\mathrm{S}^r(U)$, $\mathrm{CS}^\alpha(U)$, and so forth.

A pseudodifferential operator (ΨDO), which for a given atlas on M has a classical symbol in the local coordinates defined by each chart is called *classical*. Let $C\ell(M,E)$ denote the algebra of classical ΨDOs acting on $C^\infty(M,E)$ and let $E\ell\ell(M,E)$ be the subalgebra of elliptic operators. For any $\alpha \in \mathbb{C}$ let $C\ell^\alpha(M,E)$, resp. $E\ell\ell^\alpha(M,E)$, denote the subset of operators in $C\ell(M,E)$, resp. $E\ell\ell(M,E)$, of order α. With $\mathbb{R}_+ = (0,\infty)$, set $E\ell\ell_{\mathrm{ord}>0}(M,E) := \bigcup_{r\in\mathbb{R}_+} E\ell\ell^r(M,E)$. For a subset I of $\mathbb{C}$ we set $C\ell^I(M,E) := \bigcup_{\alpha\in I} C\ell^\alpha(M,E)$.

1.2. The noncommutative residue and the canonical trace

The local residue density on polyhomogeneous symbols acts as an obstruction to the finite part integral of a classical symbol defining a global density on M and measures the anomalous contribution to the Laurent coefficients at the poles of the finite part integral when evaluated on holomorphic families of symbols.

Definition 1. Given an open subset $U \subset \mathbb{R}^n$ and a point $x \in U$, the local noncommutative residue is defined for $\sigma \in \mathrm{CS}^\alpha(U,V)$ by

$$\mathrm{res}_x(\sigma) = \int_{S_x^*U} \mathrm{tr}_x(\sigma_{-n}(x,\xi))\, \bar{d}_S\xi,$$

where $\bar{d}_S\xi := (2\pi)^{-n}\, d_S\xi$ with $d_S\xi$ the sphere measure on S_x^*U, the unit sphere in the cotangent space T_x^*U to U at point x.

Guillemin [13] and Wodzicki [33] showed the following remarkable property.

Proposition 1. *Let $A \in C\ell^\alpha(M,E)$ be a classical ΨDO represented in a local coordinate chart U by $\sigma \in \mathrm{CS}^\alpha(U,V)$. Then $\mathrm{res}_x(\sigma)\, dx$ determines a global density on M which defines the projectively unique trace on $C\ell(M,E)$:*

$$\mathrm{res}(A) := \int_M \mathrm{res}_x(\sigma)\, \mathrm{d}x = \int_M \mathrm{d}x \int_{S_x^*M} \mathrm{tr}_x(\sigma_{-n}(x,\xi))\, \bar{d}_S\xi, \tag{3}$$

known as the noncommutative residue (and also called the Guillemin–Wodzicki residue).

The terminology refers to the trace property; if the manifold M is connected and has dimension larger than 1, then up to a scalar multiple, Eq. (3) defines on $C\ell(M,E)$ the unique linear functional vanishing on commutators

$$\mathrm{res}([A,B]) = 0, \quad A, B \in C\ell(M,E).$$

It also follows from its definition that the residue trace vanishes on operators of order $<-n$ and on non-integer order operators.

On the other hand, it was observed by Kontsevich and Vishik [17] that the usual L^2-trace on ΨDOs of real order $<-n$ extends to a functional on the space $C\ell^{\mathbb{C}\setminus\mathbb{Z}}(M,E)$ of ΨDOs of non-integer order and vanishes on commutators of non-integer order. This functional uses the following extension of the ordinary Lebesgue integral.

Definition 2. The finite-part integral of $\sigma(x,\cdot)$ with $\sigma \in \mathrm{CS}^{\alpha}(U)$ and $x \in U$ is defined as the constant term in the asymptotic expansion when $R \to \infty$ of $\int_{B_x^*(0,R)} \sigma(x,\xi)\, đ\xi$:

$$-\!\!\!\!\!\!\int_{T_x^*U} \sigma(x,\xi)\, đ\xi := \mathrm{fp}_{R\to\infty} \int_{B_x^*(0,R)} \sigma(x,\xi)\, đ\xi. \tag{4}$$

Here $B_x^*(0,R)$ is the cotangent ball of radius R in T_x^*U and $đ\xi = \frac{1}{(2\pi)^n}\, \mathrm{d}\xi$ the normalised Lebesgue measure on T_x^*U.

Whenever a classical pseudodifferential operator A has non integer order, so has its symbol σ_A, and $(-\!\!\!\!\!\int_{T_x^*U} \mathrm{tr}_x(\sigma_A(x,\xi))đ\xi)\, \mathrm{d}x$ defines a global density. Here tr_x denotes the fibrewise trace over $x \in M$. The canonical trace can therefore be defined on $C\ell^{\mathbb{C}\backslash\mathbb{Z}}(M,E)$ without ambiguity.

Definition 3. For a pseudodifferential operator $A \in C\ell^{\mathbb{C}\backslash\mathbb{Z}}(M,E)$ the canonical trace is defined by

$$\mathrm{TR}(A) := \int_M \mathrm{d}x -\!\!\!\!\!\!\int_{T_x^*U} \mathrm{tr}_x(\sigma_A(x,\xi))đ\xi.$$

It coincides with the usual trace on ΨDOs of order $<-n$.

On commutators the canonical trace has the following vanishing property, providing some justification for its name.

Proposition 2. *Let $A \in C\ell^{a,k}(M,E)$, $B \in C\ell^{b,l}(M,E)$. If $\alpha+\beta \notin [-n,\infty)\cap\mathbb{Z}$, then the canonical trace is defined on the commutator $[A,B]$ and is equal to zero,*

$$\mathrm{TR}([A,B]) = 0.$$

1.3. Cauchy integrals

An operator $Q \in E\ell\ell(M,E)$ of positive order is called *admissible* if there is a proper subsector of $\mathbb{C}$ with vertex 0 which contains the spectrum of the leading symbol $\sigma_L(Q)$ of Q. Then there is a half line $L_\phi = \{r\mathrm{e}^{\mathrm{i}\phi}, r > 0\}$ (a spectral cut) with vertex 0 and determined by an Agmon angle ϕ which does not intersect the spectrum of Q. Let $E\ell\ell^{\mathrm{adm}}_{\mathrm{ord}>0}(M,E)$ denote the subset of admissible operators in $E\ell\ell(M,E)$ with positive order.

Let $Q \in E\ell\ell^{\mathrm{adm}}_{\mathrm{ord}>0}(M,E)$ with spectral cut L_ϕ. Let f be a complex valued continuous function on the contour C_ϕ around the spectrum of Q and such that $\rho \mapsto \frac{f(\rho \mathrm{e}^{\mathrm{i}\phi})}{\rho}$ lies in $L^1(]1,\infty[)$.

Recall that an operator $R \in C\ell(M,E)$ of order r acting on smooth sections of $\pi : E \to M$ induces a bounded map from $H^s(M,E)$ to $H^{s-r}(M,E)$ for any $s \in \mathbb{R}$. For $R \in C\ell(M,E)$ and $s,t \in \mathbb{R}$ we define the operator norms (whenever they are finite)

$$\|R\|_{s,t} = \sup_{u\neq 0} \frac{\|Ru\|_t}{\|u\|_s}, \qquad \|R\|^{(s)} := \|R\|_{s,s}.$$

It follows from the theory of elliptic operators on closed manifolds that in a neighborhood of infinity (see [8] Lemma 1.7.3):

$$\|R(\lambda,Q)\|^{(s)} = O(|\lambda|^{-1}), \tag{5}$$

i.e.

$$\forall s \in \mathbb{R}, \quad \exists c(s) \in \mathbb{R}^+, \exists C_s \in \mathbb{R}^+ \text{ such that } \|R(\lambda,Q)\|^{(s)} \le C_s.$$

On the other hand, Eq. (5) and the assumption on f imply that the Cauchy integral

$$f(Q) = \frac{1}{2\mathrm{i}\pi} \int_{C_\phi} f(\lambda) R(\lambda,Q)\, \mathrm{d}\lambda$$

converges in each Sobolev norm $\|\cdot\|^{(s)}$. Here $C_\phi = C_{1,\phi,r} \cup C_{2,\phi,r} \cup C_{3,\phi,r}$. The positive real number r is chosen sufficiently small and $C_{1,\theta,r} = \{\lambda = |\lambda|\mathrm{e}^{\mathrm{i}\phi} \mid +\infty > |\lambda| \ge r\}$, $C_{2,\phi,r} = \{\lambda = r\mathrm{e}^{\mathrm{i}\psi} \mid \phi \ge \psi \ge \phi - 2\pi\}$ and

$C_{3,\phi,r} = \{\lambda = |\lambda| e^{i(\phi-2\pi)} \mid r \leq |\lambda| < +\infty\}$. Here $\lambda^z = \exp(z \log \lambda)$ where $\log \lambda = \log |\lambda| + i\phi$ on $C_{1,\phi,r}$ and $\log \lambda = \log |\lambda| + i(\theta - 2\pi)$ on $C_{3,\phi,r}$.

Remark 1. The definition of $f(Q)$ depends in general on the choice of ϕ and should therefore carry a subscript ϕ writing $f_\phi(Q)$; when there is no ambiguity in the choice of spectral cut, we may omit it in order to simplify notations.

Example 1. The function $f(z) = \lambda^{-z}$ with $\mathrm{Re}\, z > 0$ satisfies the integrability assumption so that we can define complex powers of Q by the Cauchy integral:

$$Q_\phi^{-z} = \frac{i}{2\pi} \int_{C_\phi} \lambda^{-z} (Q - \lambda I)^{-1} d\lambda \tag{6}$$

which converges in norm $\|\cdot\|^{(s)}$ for any $s \in \mathbb{R}$.

For $k \in \mathbb{N}$ the complex power Q^{-z} is then extended to the half plane $\mathrm{Re}\, z > -k$ via the relation [30] (see also [31])

$$Q^k Q_\phi^{-z-k} = Q_\phi^{-z}.$$

For $z = 0$

$$Q_\phi^0 = I - \Pi_Q$$

where Π_Q is the smoothing operator projection

$$\Pi_Q = \frac{i}{2\pi} \int_{C_0} (Q - \lambda I)^{-1} d\lambda$$

with C_0 a contour containing the origin but no other element of $\mathrm{spec}(Q)$, with range the generalised kernel $\{\psi \in C^\infty(M, E) \mid Q^N \psi = 0 \text{ for some } N \in \mathbb{N}\}$ of Q. (See [33], presented recently in [25].)

Example 2. The function $f(\lambda) = e^{-t\lambda}$ on $\mathbb{R}^+$ also has the required integrality property on C_0. Thus, if Q is a non negative self-adjoint operator, we can define the corresponding heat operator by a Cauchy integral

$$e^{-tQ} = \frac{1}{2i\pi} \int_{C_0} e^{-\lambda t} R(\lambda, Q) d\lambda$$

for any $t > 0$.

2. Quantised Cauchy integrals; first properties

The following presentation is inspired by [28] from which we borrow the notation θ. We introduce a quantised resolvent associated with a classical pseudodifferential operator Q and describe it perturbatively. It acts on the space of chains $C\ell_\bullet(M, E) = \bigotimes^{\bullet+1} C\ell(M, E)$ built from the algebra $C\ell(M, E)$ and coincides with the ordinary resolvent on $\mathcal{C}_0(M, E) = C\ell(M, E)$. This quantised resolvent leads to quantised functionals $f_\bullet(Q)$ of Q obtained from a perturbative expansion in θ of $f_\bullet(Q + \theta)$.

2.1. Quantised resolvents

For any λ outside the spectrum of some operator $Q \in C\ell(M, E)$, the resolvent $R(\lambda, Q) := (\lambda - Q)^{-1}$ is well defined. Let $\theta : C\ell(M, E) \to C\ell(M, E)$ be the identity map which we use as an insertion operator then $Q + \theta(A) = Q + A$. We have $(\lambda - Q)(\lambda - Q - \theta)^{-1} = (\lambda - Q - \theta)(\lambda - Q - \theta)^{-1} + \theta(\lambda - Q - \theta)^{-1} = 1 + \theta(\lambda - Q - \theta)^{-1}$, from which it follows that

$$(\lambda - Q - \theta)^{-1} = (\lambda - Q)^{-1} + (\lambda - Q)^{-1} \theta (\lambda - Q - \theta)^{-1}.$$

By induction we get

$$(\lambda - Q - \theta)^{-1} = \sum_{k=0}^{n} (\lambda - Q)^{-1} \theta (\lambda - Q)^{-1} \cdots \theta (\lambda - Q)^{-1} \theta (\lambda - Q)^{-1} + S_n(Q, \theta, \lambda),$$

where S_n vanishes on $\mathcal{C}_k(M,E)$ for $k \le n$. Here $\theta(\lambda-Q)^{-1}$ arises k times in the k-th term of the sum and by convention the $k=0$ term reduces to $(\lambda-Q)^{-1}$.

On each $\mathcal{C}_n(M,E)$ the expression $(\lambda-Q-\theta)^{-1}$ therefore coincides with $\sum_{k=0}^{n}(\lambda-Q)^{-1}\theta(\lambda-Q)^{-1}\cdots(\lambda-Q)^{-1}\theta(\lambda-Q)^{-1}$, which leads to the following definition.

Definition 4. From the resolvent $R(\lambda,Q) := (\lambda-Q)^{-1}$ of Q one defines the quantised resolvent of Q on $\mathcal{C}_\bullet(M,E)$ by[6]

$$\begin{aligned} R_\bullet(\lambda, Q+\theta) &:= (\lambda-(Q+\theta))^{-1} \\ &= \sum_{n=0}^{\infty} R(\lambda,Q)\,\theta\cdots R(\lambda,Q)\,\theta\, R(\lambda,Q) \quad (\theta \text{ arises } n \text{ times}). \end{aligned}$$

It induces a map on $\mathcal{C}_\bullet(M,E)$ defined by $\mathcal{R}_0(\lambda,Q) = R(\lambda,Q)$ and for any positive integer n by

$$\begin{aligned} R_n(\lambda, Q+\theta): \quad \mathcal{C}_{n-1}(M,E) &\to C\ell(M,E) \\ A_1\otimes\cdots\otimes A_n &\mapsto R(\lambda,Q)A_1\cdots R(\lambda,Q)A_n\, R(\lambda,Q). \end{aligned}$$

Remark 2. One can replace θ by some expression $h(\theta)$ in which case the A_i's would be replaced by $h(A_i)$'s. As mentioned in Remark 7, we need to choose $h(A_i) = [X, A_i]$ to set up a relationship between the constructions to follow and the construction of the Chern character in cyclic cohomology.

Let us introduce some notation which will be useful for what follows.

Definition 5. For A in $Cl(M,E)$, we set $A_Q^{(0)} = A$ and for any $j\in\mathbb{N}$:

$$A_Q^{(j)} := \mathrm{ad}_Q^j(A), \quad \text{where } \mathrm{ad}_Q(B) = [Q,B],$$

so that $A_Q^{(j+1)} = \mathrm{ad}_Q(A^{(j)}) = [Q, A^{(j)}]$.

We shall often drop the subscript Q, writing $A^{(j)}$ instead of $A_Q^{(j)}$.

The subsequent observation explains the reason for choosing the leading symbol of Q scalar in what follows.

Remark 3. If Q has scalar leading symbol then $A^{(j)}$ has order $a + j(q-1)$ where a denotes the order of A and q the order of Q.

We introduce further notations borrowed from [14]. Let $T \in C\ell(M,E)$ and T_k, $k\in\mathbb{N}$ be operators in $C\ell(M,E)$ with decreasing order in k. Then

$$T \simeq \sum_{k\ge 0} T_k \Longleftrightarrow \forall N\in\mathbb{N}, \exists K(N) \quad T - \sum_{k=0}^{K(N)} T_k \in C\ell^{-N}(M,E). \tag{7}$$

The following result that we quote from [14] (see the proof of Proposition 4.14) is a cornerstone to proving the existence of the quantised weighted traces as well as their locality.

Lemma 1 (*See [14] Lemma 4.20*). *Let $Q\in C\ell(M,E)$ be an admissible elliptic operator with spectral cut L_ϕ and scalar leading symbol. For $\lambda\in C_\phi$ and for any $A\in C\ell(M,E)$ and any non negative integer h*

$$(\lambda-Q)^{-h}A \simeq \sum_{k\ge 0} \frac{(h+k-1)!}{(h-1)!k!} A^{(k)} (\lambda-Q)^{-h-k}.$$

[6] Although a priori infinite, as we saw previously, this sum is in fact a finite sum on each of the $\mathcal{C}_n(M,E)$.

Proof. The case $h = 1$ follows from iterating the following identities

$$\begin{aligned}
[(\lambda - Q)^{-1}, A] &= (\lambda - Q)^{-1}\,[Q, A]\,(\lambda - Q)^{-1} \\
&= [(\lambda - Q)^{-1}, [Q, A]]\,(\lambda - Q)^{-1} + [Q, A]\,(\lambda - Q)^{-2} \\
&= [Q, A](\lambda - Q)^{-2} + A^{(2)}\,(\lambda - Q)^{-3} + [(\lambda - Q)^{-1}, A^{(2)}](\lambda - Q)^{-1} \\
&= [Q, A](\lambda - Q)^{-2} + A^{(2)}\,(\lambda - Q)^{-3} + A^{(3)}(\lambda - Q)^{-4} + (\lambda - Q)^{-1}\,A^{(4)}\,(\lambda - Q)^{-1} \\
&= \cdots \\
&\simeq \sum_{k\geq 1} A^{(k)}(\lambda - Q)^{-(k+1)}.
\end{aligned}$$

The general case $h > 1$ then follows using a Cauchy integral:

$$L^{-h} = \frac{1}{2i\pi}\int_{C_\phi} \mu^{-h}(\mu - L)^{-1}d\mu$$

applied to $L := \lambda - Q$ combined with integration by parts. Namely,

$$\begin{aligned}
[(\lambda - Q)^{-h}, A] &= \frac{1}{2i\pi}\int_{C_\phi} \mu^{-h}[(\mu - L)^{-1}, A]d\mu \\
&\simeq \frac{1}{2i\pi}\int_{C_\phi} \mu^{-h}\sum_{k\geq 1}(-1)^k A^{(k)}(\mu - L)^{-(k+1)}\,d\mu \\
&\simeq \frac{1}{2i\pi}\sum_{k\geq 1}(-1)^k A^{(k)}\int_{C_\phi} \mu^{-h}(\mu - L)^{-(k+1)}\,d\mu \\
&\simeq \frac{1}{2i\pi}\sum_{k\geq 1}\frac{(h+k-1)!}{(h-1)!k!}A^{(k)}\int_{C_\phi} \mu^{-h-k}(\mu - L)^{-1}\,d\mu \\
&\simeq \sum_{k\geq 1}\frac{(h+k-1)!}{(h-1)!k!}A^{(k)}(\lambda - Q)^{-h-k},
\end{aligned}$$

where we have used the fact that $\mathrm{ad}_L^j(A) = (-1)^j A^{(j)}$, hence the result. □

2.2. Cauchy integrals on higher order PDO chains

Let $Q \in E\ell\ell^{\mathrm{adm}}_{\mathrm{ord}>0}(M, E)$ have spectral cut L_ϕ and positive order q. In a similar way as we defined $f(Q)$ we define the quantised version $f_\bullet(Q+\theta)$ via a Cauchy integral using the quantised resolvent. It is obtained from a perturbative expansion in θ of $f_\theta(Q+\theta)$ which boils down to a perturbative expansion in θ of $R(\lambda, Q+\theta)$ inside the Cauchy integral.

Proposition 3. *Let f be a complex valued function defined on the contour $C_\phi \subset \mathbb{C}$ around the spectrum of Q and such that $\rho \mapsto \frac{f(\rho e^{i\phi})}{\rho^j}$ lies in $L^1(]1,\infty[)$ for any $j \in \mathbb{N}$. Then, for any integer n and for any $A_1, \ldots, A_n \in C\ell(M, E)$ the Cauchy integral*

$$\frac{1}{2i\pi}\int_{C_\phi} f(\lambda)R_n(\lambda, Q+\theta)(A_1, \ldots, A_n)d\lambda = \frac{1}{2i\pi}\int f(\lambda)R(\lambda, Q)A_1\cdots A_{n-1}R(\lambda, Q)A_nR(\lambda, Q)d\lambda$$

converges in any Sobolev norm $\|\cdot\|^{(s)}$, $s \in \mathbb{R}$.

Proof. Since $\|A\,B\|^{(s)} \leq \|A\|^{(s)}\,\|B\|^{(s)}$, for any $A_1, \ldots, A_n \in C\ell(M, E)$ of order $a_1, \ldots, a_n$ respectively,

$$\|R(\lambda, Q)A_1\cdots R(\lambda, Q)A_nR(\lambda, Q)\|^{(s)} \leq \prod_{i=1}^n \|A_i\|^{(s)}(\|R(\lambda, Q)\|^{(s)})^{n+1} = O(\lambda^{-(n+1)}).$$

Under the assumptions on f, the convergence in any Sobolev norm $\|\cdot\|^{(s)}$, $s \in \mathbb{R}$ of the Cauchy integral then follows. □

We therefore set the following definition:

Definition 6. Let f be a complex valued function defined on the contour $C_\phi \subset \mathbb{C}$ around the spectrum of Q and such that $\rho \mapsto \frac{f(\rho e^{i\phi})}{\rho^j}$ lies in $L^1(]1,\infty[)$ for any $j \in \mathbb{N}$. The quantised Cauchy integral is defined by:

$$\begin{aligned} f_\bullet(Q+\theta) &:= \frac{1}{2i\pi}\int_{C_\phi} f(\lambda) R_\bullet(\lambda, Q+\theta)\, d\lambda \\ &= \frac{1}{2i\pi}\int f(\lambda) R(\lambda, Q)\theta \cdots \theta R(\lambda, Q)\theta R(\lambda, Q) \end{aligned}$$

where the expression $\theta R(\lambda, Q)$ arises n times. In particular, $f_0(Q) = f(Q)$ and for any positive integer n we have

$$\begin{aligned} f_n(Q+\theta)(A_1 \otimes \cdots \otimes A_n) &= \frac{1}{2i\pi}\int_{C_\phi} d\lambda f(\lambda) R_n(\lambda, Q+\theta)(A_1 \otimes \cdots \otimes A_n) \\ &= \frac{1}{2i\pi}\int_{C_\phi} d\lambda f(\lambda)\, R(\lambda, Q)A_1 \cdots R(\lambda, Q)A_n R(\lambda, Q) \end{aligned}$$

for all $A_1 \otimes \cdots \otimes A_n \in \mathcal{C}_n(M, E)$.

Remark 4. $f_\bullet(Q+\theta)$ induces a map:

$$\begin{aligned} \mathcal{C}_\bullet(M, E) &\to C\ell(M, E) \\ A_0 \otimes \cdots \otimes A_n &\mapsto (\theta f_n(Q+\theta))(A_0 \otimes \cdots \otimes A_n) \\ &= A_0 f_n(Q+\theta)(A_1 \otimes \cdots \otimes A_n). \end{aligned}$$

Example 3. For $f(\lambda) = \lambda^{-z}$, and $\mathrm{Re}(z) > 0$ the assumptions of the proposition are satisfied so that $(Q+\theta)^{-z}$ is well defined. This definition extends to $\mathrm{Re}(z) > -k$ as usual.

We now slightly weaken the definition $\simeq$ introduced above. Let $T \in C\ell(M, E)$ and $T_k, k \in \mathbb{N}$ be operators in $C\ell(M, E)$ with decreasing order in k. Then

$$T \sim \sum_{k\geq 0} T_k \Longleftrightarrow T C^{-1} \simeq \sum_{k\geq 0} T_k C^{-1} \tag{8}$$

for some invertible $C \in C\ell(M, E)$.

Theorem 1. *Let f be a complex valued function defined on the contour $C_\phi \subset \mathbb{C}$ around the spectrum of Q such that $\rho \mapsto \frac{f(\rho e^{i\phi})}{\rho^j}$ lies in $L^1(]1,\infty[)$ for any $j \in \mathbb{N}$. In particular, for any $n \in \mathbb{N}$, and for any multiindex $(k_1, \ldots, k_n) \in \mathbb{N}^n$, $f^{(|k|+n)}(Q)$ is defined by a Cauchy integral:*

$$f^{(|k|+n)}(Q) = \frac{(|k|+n)!}{2i\pi}\int_{C_\phi} f(\lambda)(\lambda - Q)^{-|k|-n-1}$$

which converges in the $\|\cdot\|^{(s)}$ norms for all $s \in \mathbb{R}$. Here $|k| = k_1 + \cdots + k_n$.

We assume that Q has scalar leading symbol, that $f^{(|k|+n)}(Q)$ lies in $C\ell(M, E)$ and that $|k|(q-1) + o(f^{(|k|+n)}(Q)C^{-1})$ decreases for some fixed invertible operator $C \in C\ell(M, E)$ as $|k|$ increases, where $o(A)$ denotes the order of A. Then for any $A_1, \ldots, A_n \in C\ell(M, E)$ the operator $A_1^{(k_1)} \cdots A_n^{(k_n)} f^{(|k|+n-1)}(Q) \in C\ell(M, E)$ has decreasing order in $|k|$ and

$$f_\bullet(Q+\theta)(A_1 \otimes \cdots \otimes A_n) \sim \sum_{|k|\geq 0} \frac{c(k)}{(|k|+n)!} A_1^{(k_1)} \cdots A_n^{(k_n)}\, f^{(|k|+n)}(Q),$$

where for a multiindex $k = (k_1, \ldots, k_n)$ and $n = 1$ we set $c(k_1) = 1$, for $n \geq 2$

$$c(k_1, \ldots, k_n) = \frac{(k_1 + \cdots + k_n + n - 1)!}{k_1! \cdots k_n!(k_1+1)(k_1+k_2+2)\cdots(k_1+\cdots+k_{n-1}+n-1)}.$$

Remark 5. When $n = 1$ we get back that $f(Q)\, A \sim \sum_{k\geq 0} \frac{A^{(k)}\, f^{(k+1)}(Q)}{(k+1)!}$.

Proof. Applying Lemma 1 to $h_i = k_1 + \cdots + k_i + i + 1$ with $i = 1, \ldots, n-1$ we find

$$\begin{aligned}
R_\bullet(\lambda, Q+\theta)(A_1 \otimes \cdots \otimes A_n) &= (\lambda - Q)^{-1} A_1 \cdots (\lambda - Q)^{-1} A_n\, (\lambda - Q)^{-1} \\
&\simeq \sum_{k_1 \geq 0} A_1^{(k_1)} (\lambda - Q)^{-2-k_1} A_2 \cdots (\lambda - Q)^{-1} A_n\, (\lambda - Q)^{-1} \\
&\simeq \sum_{k_1 \geq 0} A_1^{(k_1)} \sum_{k_2 \geq 0} \frac{(k_1 + k_2 + 1)!}{(k_1 + 1)! k_2!} A_2^{(k_2)} (\lambda - Q)^{-3-k_1-k_2} \\
&\quad \times A_3 \cdots (\lambda - Q)^{-1} A_n\, (\lambda - Q)^{-1} \\
&\simeq \sum_{k_1 \geq 0} A_1^{(k_1)} \sum_{k_2 \geq 0} \frac{(k_1 + k_2 + 1)!}{(k_1 + 1)! k_2!} A_2^{(k_2)} \sum_{k_3 \geq 0} \frac{(k_1 + k_2 + k_3 + 2)!}{(k_1 + k_2 + 2)! k_3!} \\
&\quad \times A_3^{(k_3)} (\lambda - Q)^{-4-k_1-k_2-k_3} A_4 \cdots (\lambda - Q)^{-1} A_n\, (\lambda - Q)^{-1} \\
&\simeq \sum_{|k| \geq 0} \frac{(k_1 + k_2 + 1)! \cdots (k_1 + k_2 + \cdots + k_n + n - 1)!}{(k_1 + 1)! \cdots k_n! (k_1 + 1)(k_1 + k_2 + 2)! \cdots (k_1 + \cdots + k_{n-1} + n - 1)!} \\
&\quad \times A_1^{(k_1)} A_2^{(k_2)} \cdots A_n^{(k_n)}\, (\lambda - Q)^{-|k|-n-1} \\
&\simeq \sum_{|k| \geq 0} c(k_1, \ldots, k_n)\, A_1^{(k_1)} A_2^{(k_2)} \cdots A_n^{(k_n)}\, (\lambda - Q)^{-|k|-n-1}.
\end{aligned}$$

It is useful to keep in mind that since Q has scalar symbol, the product $A_1^{(k_1)} A_2^{(k_2)} \cdots A_n^{(k_n)}$ has order $|a| + |k|(q-1)$ where $|a|$ is the total order of the product $A_1 \cdots A_n$ so that $A_1^{(k_1)} A_2^{(k_2)} \cdots A_n^{(k_n)}\, (\lambda - Q)^{-|k|-n-1}$ has order $|a| + |k|\,(q-1) - q\,(|k| + n + 1) = |a| - |k| - q(n+1)$ which decreases as $|k|$ grows.

On the other hand, integrating by parts yields

$$\begin{aligned}
\frac{1}{2i\pi} \int f(\lambda)(\lambda - Q)^{-(|k|+n+1)}\, d\lambda &= \frac{1}{2i\pi\,(|k|+n)!} \int f^{(|k|+n)}(\lambda)(\lambda - Q)^{-1}\, d\lambda \\
&= \frac{1}{(|k|+n)!}\, f^{(|k|+n)}(Q).
\end{aligned}$$

For any operator C chosen as in the statement of the proposition, this yields

$$\begin{aligned}
f_n(Q+\theta)(A_1 \otimes \cdots \otimes A_n) C^{-1} &= \frac{1}{2i\pi} \int f(\lambda) R_n(\lambda, Q+\theta)(A_1 \otimes \cdots \otimes A_n)\, d\lambda \cdot C^{-1} \\
&\simeq \sum_{|k| \geq 0} \frac{c(k)}{(|k|+n)!} A_1^{(k_1)} \cdots A_n^{(k_n)}\, \frac{1}{2i\pi} \int f(\lambda)(\lambda - Q)^{-|k|-n-1}\, d\lambda\, \cdot C^{-1} \\
&\simeq \sum_{|k| \geq 0} \frac{c(k)}{(|k|+n)!}\, A_1^{(k_1)} \cdots A_n^{(k_n)}\, f^{(|k|+n)}(Q)\, C^{-1}.
\end{aligned}$$

Since by assumption the order of $A_1^{(k_1)} \cdots A_n^{(k_n)}\, f^{(|k|+n)}(Q)\, C^{-1}$ decreases as $|k|$ grows, it follows that

$$f_n(Q+\theta)(A_1 \otimes \cdots \otimes A_n) \sim \sum_{|k| \geq 0} \frac{c(k)}{(|k|+n)!}\, A_1^{(k_1)} \cdots A_n^{(k_n)}\, f^{(|k|+n)}(Q),$$

which ends the proof of the theorem. □

Corollary 1. *Let $Q \in C\ell(M, E)$ be an admissible elliptic operator with spectral cut ϕ and scalar leading symbol. The quantised complex power of Q is well defined for* $\mathrm{Re}(z) > 0$ *and for any $A_1, \ldots, A_n \in C\ell(M, E)$ we have*

$$(Q+\theta)^{-z}\, (A_1 \otimes \cdots \otimes A_n) \sim \sum_{|k| \geq 0} \frac{(-1)^{|k|+n} c(k)\, \Gamma(z + |k| + n)}{\Gamma(z)(|k|+n)!}\, A_1^{(k_1)} \cdots A_n^{(k_n)}\, Q^{-z-|k|-n}.$$

Remark 6. The operator $A_0(Q+\theta)^{-z}(A_1\otimes\cdots\otimes A_n)$ coincides with the operator involved in the definition of $\langle A_0,\ldots,A_n\rangle_z$ in [14] (see paragraph 4.3) up to a sign and a multiplicative factor $\Gamma(z)$.

Proof. For any complex number z with positive real part the map $f(\lambda)=\lambda^{-z}$ satisfies the assumptions of Theorem 1 (here we take $C=1$) so that the quantised complex power of Q reads

$$(Q+\theta)^{-z}(A_1\otimes\cdots\otimes A_n)\sim\sum_{k\geq 0}\frac{c(k)(-z)(-z-1)\cdots(-z-(|k|+n-1))}{(|k|+n)!}A_1^{(k_1)}\cdots A_n^{(k_n)}Q^{-z-(|k|+n)}$$

$$\sim\sum_{|k|\geq 0}\frac{(-1)^{|k|+n}\,c(k)(z+|k|+n-1)\cdots(z+1)z}{(|k|+n)!}A_1^{(k_1)}\cdots A_n^{(k_n)}Q^{-z-(|k|+n)}$$

$$\sim\sum_{|k|\geq 0}\frac{(-1)^{|k|+n}\,c(k)\Gamma(z+|k|+n)}{\Gamma(z)(|k|+n)!}A_1^{(k_1)}\cdots A_n^{(k_n)}\,Q^{-z-(|k|+n)},$$

which proves the lemma. □

Corollary 2. *Let $Q\in C\ell(M,E)$ be an elliptic operator with positive order and positive leading symbol, for any $A_1,\ldots,A_n\in C\ell(M,E)$ we have for any $\varepsilon>0$*

$$\mathrm{e}^{-\varepsilon(Q+\theta)}(A_1\otimes\cdots\otimes A_n)=(-\varepsilon)^n\int_{\Delta_n}\mathrm{e}^{-\varepsilon u_0 Q}A_1\cdots\mathrm{e}^{-\varepsilon u_{n-1}Q}A_n\mathrm{e}^{-\varepsilon u_n Q}\mathrm{d}u_1\cdots\mathrm{d}u_n$$

$$\sim\sum_{|k|\geq 0}\frac{(-1)^{|k|+n}c(k)\varepsilon^{|k|+n}}{(|k|+n)!}A_1^{(k_1)}\cdots A_n^{(k_n)}\mathrm{e}^{-\varepsilon Q},$$

where $\Delta_n:=\{(u_0,\ldots,u_n),u_i\geq 0,\sum_{i=0}^n u_i=1\}$ is the unit simplex.

Remark 7.
- The operator $A_0\mathrm{e}^{-\varepsilon(Q+\theta)}(A_1\otimes\cdots\otimes A_n)$ coincides with the operators involved in JLO traces used in [14] (see Appendix A) up to a multiplicative factor $(-1)^n\varepsilon^{\frac{n}{2}}$.
- If M is a spin manifold and E the spinor bundle on M, letting $Q=D^2$ with D the Dirac operator, following [28], one can treat $\theta+\sigma D$ as a superconnection form where σ is the grading. The curvature is then given by $R=Q+\sigma[D,\theta]$ and (see formula (8.1) in [28])

$$\mathrm{e}^{Q+\sigma[D,\theta]}=\sum_{n\geq 0}\int_{\Delta_n}\mathrm{e}^{u_0 Q}\sigma[D,\theta]\cdots\mathrm{e}^{u_{n-1}Q}\sigma[D,\theta]\mathrm{e}^{u_n Q}\,\mathrm{d}u_1\cdots\mathrm{d}u_n$$

  can be seen as a quantised heat operator up to the fact that one has replaced θ by $\sigma[D,\theta]$.

Proof. Applying Theorem 1 to $f(\lambda)=\mathrm{e}^{-\varepsilon\lambda}$ (here we take $C=\mathrm{e}^{-\varepsilon Q}$), the quantised heat operator of Q yields

$$\mathrm{e}^{-\varepsilon(Q+\theta)}(A_1\otimes\cdots\otimes A_n)\sim\sum_{|k|\geq 0}\frac{(-1)^{|k|+n}c(k)\,\varepsilon^{|k|+n}}{(|k|+n)!}A_1^{(k_1)}\cdots A_n^{(k_n)}\mathrm{e}^{-\varepsilon Q}.$$

As for the second statement in the corollary, clearly,

$$\mathrm{e}^{-\varepsilon(Q+\theta)}(A_1\otimes\cdots\otimes A_n)=(-\varepsilon)^n\mathrm{e}^{-\varepsilon Q+\theta}(A_1\otimes\cdots\otimes A_n)$$

so it suffices to prove that

$$\mathrm{e}^{Q+\theta}(A_1\otimes\cdots\otimes A_n)=\int_{\Delta_n}\mathrm{e}^{u_0 Q}A_1\cdots\mathrm{e}^{u_{n-1}Q}A_n\mathrm{e}^{u_n Q}\mathrm{d}u.$$

and then substitute $-\varepsilon Q$ for Q.

The following derivation is taken from [14]. Using the Cauchy integral as a contour integral along the imaginary axis we write

$$\mathrm{e}^{Q+\theta}(A_1\otimes\cdots\otimes A_n)=\frac{1}{2\mathrm{i}\pi}\int\mathrm{e}^{\lambda}(\lambda-Q)^{-1}A_1\cdots(\lambda-Q)^{-1}A_n(\lambda-Q)^{-1}\mathrm{d}\lambda$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\nu}(\mathrm{i}\nu - Q)^{-1}A_1\cdots(\mathrm{i}\nu - Q)^{-1}A_n(\mathrm{i}\nu - Q)^{-1}\,\mathrm{d}\lambda \\
&= \lim_{\delta\to 0}\frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\nu}\,\hat{f}_0^{\delta}(\nu)\cdots\hat{f}_n^{\delta}(\nu)\mathrm{d}\nu \\
&= \lim_{\delta\to 0}(f_0^{\delta}\star\cdots\star f_n^{\delta})(1) \\
&= \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} g_0(1-u_1)g_1(u_1-u_2)\cdots g_{n-1}(u_{n-1}-u_n)g_n(u_n)\mathrm{d}u_1\cdots\mathrm{d}u_n \\
&= \int_{\Delta_n} \mathrm{e}^{u_0 Q}A_1\cdots A_{n-1}\mathrm{e}^{u_{n-1}Q}A_n\mathrm{e}^{u_n Q}\mathrm{d}u_1\cdots\mathrm{d}u_n,
\end{aligned}
$$

where f_j^{δ} is the convolution of $g_j(t) = A_j\mathrm{e}^{t\,Q}$ (here $A_0 = 1$) if $t \geq 0$ and $g_j(t) = 0$ if $t < 0$ with $\delta^{-1}\phi(\delta^{-1}\cdot)$ some compactly supported bump function. □

2.3. Transformation under the adjoint action of $C\ell(M,E)$

An operator $C \in C\ell(M,E)$ induces an adjoint action:

$$
\begin{array}{rcl}
C\ell(M,E) & \to & C\ell(M,E) \\
A & \mapsto & \mathrm{ad}_C(A) := [C,A].
\end{array}
$$

The following proposition shows how a quantised resolvent of level n transforms to a quantised resolvent of level $n+1$ under the adjoint action.

Proposition 4. *For any $A_1,\ldots,A_n \in C\ell(M,E)$ and any $C \in C\ell(M,E)$, for any $Q \in C\ell(M,E)$ and any λ outside the spectrum of Q,*

$$
\begin{aligned}
(\mathrm{ad}_C\, R_n(\lambda, Q+\theta))(A_1\otimes\cdots\otimes A_n) &:= \mathrm{ad}_C(R_n(\lambda, Q+\theta)(A_1\otimes\cdots\otimes A_n)) \\
&\quad - \sum_{i=1}^{n} R_n(\lambda, Q+\theta)(A_1\otimes\cdots\otimes \mathrm{ad}_C A_i\otimes\cdots\otimes A_n) \\
&= \sum_{j=0}^{n} R_{n+1}(\lambda, Q+\theta)(A_1\otimes\cdots\otimes A_j\otimes \mathrm{ad}_C(Q)\otimes A_{j+1}\otimes\cdots\otimes A_n).
\end{aligned}
$$

Proof. Since the adjoint action is a derivation $\mathrm{ad}_C(AB) = \mathrm{ad}_C(A)B + A\mathrm{ad}_C(B)$ and since

$$
\mathrm{ad}_C R(\lambda, Q) = R(\lambda, Q)\mathrm{ad}_C Q R(\lambda, Q),
$$

we have

$$
\begin{aligned}
&(\mathrm{ad}_C R_n(\lambda, Q+\theta))(A_1\otimes\cdots\otimes A_n) \\
&\quad = \sum_{j=0}^{n} R(\lambda,Q)A_1R(\lambda,Q)\cdots A_j\mathrm{ad}_C R(\lambda,Q)A_{j+1}R(\lambda,Q)\cdots R(\lambda,Q)A_nR(\lambda,Q) \\
&\quad = \sum_{j=0}^{n} R(\lambda,Q)A_1R(\lambda,Q)\cdots A_jR(\lambda,Q)\mathrm{ad}_C(Q)R(\lambda,Q)A_{j+1}R(\lambda,Q)\cdots R(\lambda,Q)A_nR(\lambda,Q),
\end{aligned}
$$

leading to the statement of the proposition. □

Performing a Cauchy integral leads to the following result.

Corollary 3. *Let $Q \in E\ell\ell^{\mathrm{adm}}_{\mathrm{ord}>0}(M,E)$ have spectral cut L_ϕ and positive order q. For any $A_1,\ldots,A_n \in C\ell(M,E)$ and any $C \in C\ell(M,E)$*

$$
(\mathrm{ad}_C\, f_n(Q+\theta))(A_1\otimes\cdots\otimes A_n) = \sum_{j=0}^{n} f_{n+1}(Q+\theta)(A_1\otimes\cdots\otimes A_j\otimes \mathrm{ad}_C(Q)\otimes A_{j+1}\otimes\cdots\otimes A_n)
$$

where f is any complex valued function defined on the contour C_ϕ around the spectrum of Q and such that $\rho \mapsto \frac{f(\rho e^{i\phi})}{\rho^j}$ lies in $L^1(]1, \infty[)$ for any $j \in \mathbb{N}$.

Combining Corollary 3 applied to $f(\lambda) = e^{-\varepsilon\lambda}$ with Corollary 2 yields back the following well-known formula.

Corollary 4. *Let $A_1, \ldots, A_n \in C\ell(M, E)$ and let $Q \in Ell(M, E)$ have positive leading symbol then for any $C \in C\ell(M, E)$ and any $\varepsilon > 0$*

$$\begin{aligned} \mathrm{ad}_C(e^{-\varepsilon(Q+\theta)})(A_1 \otimes \cdots \otimes A_n) &= (-\varepsilon)^n \mathrm{ad}_C\left[\int_{\Delta_n} \mathrm{d}u e^{-\varepsilon u_0 Q} A_1 \cdots e^{-\varepsilon u_n Q} A_n\right] \\ &= (-\varepsilon)^n \sum_{j=1}^{n+1} \int_{\Delta_n} \mathrm{d}u e^{-\varepsilon u_0 Q} A_1 \cdots A_{j-1} e^{-\varepsilon u_{j-1} Q} \mathrm{ad}_C Q e^{-\varepsilon u_j Q} \\ &\quad \times A_j e^{-\varepsilon u_{j+1} Q} \cdots e^{-\varepsilon u_n Q} A_n. \end{aligned}$$

2.4. Varying the operator Q

The following elementary result will be useful to control the variation of quantised weighted traces when the weight Q varies.

Proposition 5. *Let $Q(b) \in C\ell(M, E)$ be a differentiable family of classical pseudodifferential operators parametrised by a manifold B and let $A_1, \ldots, A_n \in C\ell(M, E)$. Given a point $b \in B$, for any λ outside the spectrum of $Q(b)$ with b in some neighborhhod of a point $b_0 \in B$,*

$$\begin{aligned} &\mathrm{d}R_n(\lambda, Q + \theta)(b_0)\,(A_1 \otimes \cdots \otimes A_n) \\ &\quad = \sum_{j=1}^{n} R_{n+1}(\lambda, Q(b_0) + \theta)(A_1 \otimes \cdots \otimes A_{j-1} \otimes \mathrm{d}Q(b_0) \otimes A_j \otimes \cdots \otimes A_n). \end{aligned}$$

Proof. This follows from the identity

$$\mathrm{d}R(\lambda, Q) = R(\lambda, Q)\mathrm{d}Q R(\lambda, Q)$$

combined with the Leibniz rule. □

Performing a Cauchy integral along the contour C_ϕ yields the following result.

Corollary 5. *Let $Q(b) \in Ell^{\mathrm{adm}}_{\mathrm{ord}>0}(M, E), b \in B$ be a differentiable family with fixed positive order q and a given common spectral cut L_ϕ. For any non negative integer n and any $A_1, \ldots, A_n \in C\ell(M, E)$,*

$$(\mathrm{d}\, f_n(Q + \theta))(A_1 \otimes \cdots \otimes A_n) = \sum_{j=1}^{n} f_{n+1}(Q + \theta)\,(A_1 \otimes \cdots \otimes A_j \otimes \mathrm{d}Q \otimes A_{j+1} \otimes \cdots \otimes A_n)$$

where f is any complex valued function defined on the contour C_ϕ around the spectrum of Q and such that $\rho \mapsto \frac{f(\rho e^{i\phi})}{\rho^j}$ lies in $L^1(]1, \infty[)$ for any $j \in \mathbb{N}$.

Remark 8. When $n = 0$ this reads

$$\mathrm{d} f(Q)(A) = f_1(Q, \theta)(A \otimes \mathrm{d}\, Q).$$

Combining Corollary 5 applied to $f(\lambda) = e^{-\varepsilon\lambda}$ with Corollary 2 yields back the following well-known formula.

Corollary 6. *Let $A_1, \ldots, A_n \in C\ell(M, E)$ and let $Q(b), b \in B$ be a differentiable family in $E\ell\ell(M, E)$ of operators with positive leading symbol parametrised by a manifold B then*

$$\begin{aligned}
&\mathrm{d}(\mathrm{e}^{-\varepsilon(Q+\theta)}(A_1 \otimes \cdots \otimes A_n)) \\
&= (-\varepsilon)^n \mathrm{d}\left[\int_{\Delta_n} \mathrm{d}u \mathrm{e}^{-\varepsilon u_0 Q} A_1 \cdots \mathrm{e}^{-\varepsilon u_n Q} A_n\right] \\
&= (-\varepsilon)^n \sum_{j=1}^{n+1} \int_{\Delta_n} \mathrm{d}u \mathrm{e}^{-\varepsilon u_0 Q} A_1 \cdots A_{j-1} \mathrm{e}^{-\varepsilon u_{j-1} Q} \mathrm{d}Q \mathrm{e}^{-\varepsilon u_j Q} A_j \mathrm{e}^{-\varepsilon u_{j+1} Q} \ldots \mathrm{e}^{-\varepsilon u_n Q} A_n.
\end{aligned}$$

3. Quantised regularisation procedures

A regularisation procedure on $C\ell(M, E)$ is a map

$$\begin{aligned}
&\mathcal{R} : C\ell(M, E) \to \mathcal{A}(W, C\ell(M, E)) \\
&A \mapsto A(w)
\end{aligned}$$

with values in the space $\mathcal{A}(W, C\ell(M, E))$ of analytic maps with values in $C\ell(M, E)$, defined on a subset $W \subset \mathbb{C}$ containing $\mathbb{R}^+$ satisfying the following requirements

1. $A(w)$ is trace-class for w with large enough real part,
2. $A(0) = A$,
3. if it is finite, the order of $A(w)$ has non vanishing derivative at 0.

We focus here on regularisation procedures of the type

$$\mathcal{R}^{Q,f}(A)(w) = \theta\, f(w)(Q)(A) = A\, f(w)(Q)$$

where $f(w)(Q) := \frac{1}{2i\pi} \int f(w)(\lambda)(\lambda - Q)^{-1}\, \mathrm{d}\lambda$ for some $Q \in C\ell(M, E)$ lies in $\mathcal{A}(W, C\ell(M, E))$.

Applying the (second) quantisation procedure described previously, we can quantise such a regularisation procedure to build

$$\mathcal{R}^{Q,f}_\bullet : \mathcal{C}_\bullet(M, E) \to \mathcal{A}(W, C\ell(M, E))$$

defined by $\theta\, \mathcal{R}^{Q,f}_0(A)(w) = A\, f(w)(Q)$ and for positive integer n by

$$\begin{aligned}
&\mathcal{R}^{Q+\theta,f}_n : \mathcal{C}_n((M, E)) \to \mathcal{A}(W, C\ell(M, E)) \\
&A_0 \otimes \cdots \otimes A_n \mapsto (w \mapsto \theta\, f_n(w)(Q + \theta)(A_0 \otimes \cdots \otimes A_n)).
\end{aligned}$$

Two well-known examples are

1. when Q is an admissible elliptic operator, the zeta regularisation defined for z in $W = \mathbb{C}$ with $f(z)(\lambda) = \lambda^{-z}$ by
$$\begin{aligned}
&\mathcal{R}^{Q,\zeta} : C\ell(M, E) \to \mathrm{Hol}(\mathbb{C}, C\ell(M, E)) \\
&A \mapsto (z \mapsto A\,(Q + \pi_Q)^{-z})
\end{aligned}$$
with Q of order $q > 0$, π_Q the orthogonal projection onto the kernel of Q.
2. When Q has positive leading symbol, the heat-kernel regularisation is defined for ε in $W = \mathbb{R}^+ \cup \{0\}$ with $f(\varepsilon)(\lambda) = \mathrm{e}^{-\varepsilon\lambda}$ by
$$\begin{aligned}
&\mathcal{R}^{Q,HK} : C\ell(M, E) \to C^\infty(\mathbb{R}^+ \cup \{0\}, C\ell(M, E)) \\
&A \mapsto (\varepsilon \mapsto A\, \mathrm{e}^{-\varepsilon Q}).
\end{aligned}$$

We investigate their quantised versions

$$\mathcal{R}^{Q,\zeta}_\bullet := \mathcal{R}^{Q+\theta,\zeta}; \qquad \mathcal{R}^{Q,HK}_\bullet := \mathcal{R}^{Q+\theta,HK}$$

and study the corresponding quantised regularised traces.

3.1. Quantised zeta regularisation

Let $Q \in C\ell(M,E)$ be an admissible elliptic operator with spectral cut ϕ with positive order q. Setting $f(z)(\lambda)=\lambda^{-z}$, the zeta regularisation $\zeta^{Q,\zeta}$ can be quantised to $\mathcal{R}_\bullet^{Q+\theta,\zeta}:\mathcal{C}_\bullet(M,E)\to \mathrm{Hol}(C\ell(M,E))$ defined by $\mathcal{R}_0^{Q+\theta,\zeta}=R^{Q,\zeta}$ and for positive integer n by

$$\begin{aligned}&\mathcal{R}_n^{Q+\theta,\zeta}:\mathcal{C}_n(M,E)\to \mathrm{Hol}(C\ell(M,E))\\ &A_0\otimes A_1\otimes\cdots\otimes A_n\mapsto (z\mapsto A_0\, f_\bullet(z)(Q+\pi_Q+\theta)(A_1\otimes\cdots\otimes A_n))\end{aligned}$$

so that

$$\begin{aligned}\mathcal{R}_n^{Q+\theta,\zeta}(A_0\otimes A_1\otimes\cdots\otimes A_n) &= A_0\,(Q+\pi_Q+\theta)^{-z}(A_1\otimes\cdots\otimes A_n)\\ &=\frac{1}{2\mathrm{i}\pi}\int \lambda^{-z}A_0(\lambda-(Q+\pi_Q+\theta))^{-1}(A_1\otimes\cdots\otimes A_n)\\ &=\frac{1}{2\mathrm{i}\pi}\int \lambda^{-z}A_0(\lambda-(Q+\pi_Q))^{-1}A_1\cdots A_n\,(\lambda-(Q+\pi_Q))^{-1}.\end{aligned}$$

For simplicity, in the following we assume that Q is invertible. The results can then be easily extended to the general case by replacing Q with $Q+\pi_Q$.

The subsequent lemma summarises a well-known result (see e.g. [13,17,18,27,33]).

Lemma 2. *Let $A\in C\ell(M,E)$ of order a and $Q\in C\ell(M,E)$ an admissible elliptic operator with positive order q. Then the map $z\mapsto \mathrm{tr}(\mathcal{R}^{Q,\zeta}(A))(z):=\mathrm{tr}(AQ^{-z})$ which is holomorphic on the half plane $\mathrm{Re}(z)>\frac{a+d}{q}$ where d is the dimension of M, extends to a meromorphic map $\mathrm{tr}(\mathcal{R}^{Q,\zeta}(A))\,|^{\mathrm{mer}}$ with simple poles. The pole at 0 is proportional to the noncommutative residue:*

$$\mathrm{Res}_{z=0}\mathrm{tr}(\mathcal{R}^{Q,\zeta}(A))\,|^{\mathrm{mer}}=\frac{1}{q}\mathrm{res}(A).$$

The finite part at $z=0$

$$\mathrm{tr}^Q(A):=\mathrm{fp}_{z=0}(\mathrm{tr}(AQ^{-z}))$$

which in general is non local, boils down to a local expression when A is a differential operator [27]:

$$\mathrm{tr}^Q(A)=-\frac{1}{q}\mathrm{res}(A\log Q).$$

Theorem 2. *Let $Q\in C\ell(M,E)$ be an elliptic operator with* scalar leading symbol. *For any $A_0,\ldots,A_n\in C\ell(M,E)$, $n>0$, there exists some $K>0$ such that the expression*

$$\mathcal{R}_n^{Q+\theta,\zeta}(A_0\otimes A_1\otimes\cdots\otimes A_n)(z)-\sum_{|k|=0}^{K}\frac{c(k)(-1)^{|k|+n}\Gamma(z+|k|+n)}{\Gamma(z)(|k|+n)!}A_0A_1^{(k_1)}\cdots A_n^{(k_n)}Q^{-z-(|k|+n)}$$

is trace-class for $\mathrm{Re}(z)>\frac{d-K+|a|}{q}-n+1$ and the map $z\mapsto \mathrm{tr}(\mathcal{R}_n^{Q+\theta,\zeta}(A_0\otimes A_1\otimes\cdots\otimes A_n)(z))$ extends to a meromorphic map $\mathrm{tr}(\mathcal{R}_n^{Q+\theta,\zeta}(A_0\otimes A_1\otimes\cdots\otimes A_n))\,|^{\mathrm{mer}}$ defined on the whole plane. Its pole at $z=0$ vanishes and its limit at $z=0$ is given by a local expression:

$$\begin{aligned}\mathrm{tr}(\mathcal{R}_n^{Q+\theta,\zeta}(A_0\otimes A_1\otimes\cdots\otimes A_n))\,|^{\mathrm{mer}}_{z=0}&=\lim_{z\to 0}\mathrm{tr}(\mathcal{R}_n^{Q+\theta,\zeta}(A_0\otimes A_1\otimes\cdots\otimes A_n))\,|^{\mathrm{mer}}\\ &=\sum_{0\le|k|\le|a|-nq+d}\frac{(-1)^{|k|+n}c(k)}{q(|k|+n)}\mathrm{res}(A_0\,A_1^{(k_1)}\cdots A_n^{(k_n)}Q^{-(|k|+n)}),\end{aligned}\tag{9}$$

where as before, d is the dimension of M.

Remark 9. Since Q has scalar leading symbol, the operator $A^{(k)}$ has order $a + k(q-1)$ where a is the order of A and the operator $A_0\, A_1^{(k_1)} \cdots A_n^{(k_n)}\, Q^{-(|k|+n)}$ has order $|a| + |k|\,(q-1) - (|k|+n)q = |a| - |k| - nq$ which decreases as $|k|$ increases. Since the noncommutative residue vanishes for operators of order $< -d$ where d is the underlying dimension of the manifold M, only a finite number ($|k| \leq |a| - nq + d$) of non vanishing residues arise in (9).

Proof. From Corollary 1 we have

$$\mathcal{R}_n^{Q+\theta,\zeta}(A_0 \otimes A_1 \otimes \cdots \otimes A_n)(z) \sim \sum_{|k|\geq 0} \frac{c(k)(-1)^{|k|+n}\,\Gamma(z+|k|+n)}{\Gamma(z)\,(|k|+n)!}\, A_0\, A_1^{(k_1)} \cdots A_n^{(k_n)}\, Q^{-z-(|k|+n)}.$$

The operator $A_0\, A_1^{(k_1)} \cdots A_n^{(k_n)}\, Q^{-(|k|+n)-z}$ of order $|a| - |k| - nq - qz$ is trace class for $\mathrm{Re}(z) > \frac{|a|-|k|+d}{q} - n$. Since the operators $A_i^{(k_i)}$ are classical its trace, which defines a holomorphic function on this half plane, extends to a meromorphic function on the whole complex plane with simple poles.

For positive integer n

$$\frac{\Gamma(z+|k|+n)}{\Gamma(z)\,(|k|+n)!} \sim_0 \frac{z}{|k|+n}$$

so that by Lemma 2,

$$\lim_{z\to 0} \frac{\Gamma(z+|k|+n)}{\Gamma(z)(|k|+n)!} \mathrm{tr}(A_0\, A_1^{(k_1)} \cdots A_n^{(k_n)} Q^{-z-(|k|+n)}) = \frac{\mathrm{res}(A_0\, A_1^{(k_1)} \cdots A_n^{(k_n)} Q^{-(|k|+n)})}{q(|k|+n)}.$$

Hence, as $z \to 0$, $\mathrm{tr}(\mathcal{R}_n^{Q+\theta,\zeta}(A_0 \otimes A_1 \otimes \cdots \otimes A_n)(z))$ converges to

$$\begin{aligned}
&\mathrm{tr}(\mathcal{R}_n^{Q+\theta,\zeta}(A_0 \otimes A_1 \otimes \cdots \otimes A_n)(z))|_{z=0} \\
&\quad = \sum_{0\leq |k| \leq |a|-nq+d} \frac{(-1)^{|k|+n}\, c(k)}{q(|k|+n)}\, \mathrm{res}(A_0\, A_1^{(k_1)} \cdots A_n^{(k_n)}\, Q^{-(|k|+n)}),
\end{aligned}$$

which is a local expression as a finite linear combination of noncommutative residues. □

Definition 7. We call the value at $z = 0$ of the meromorphic extension

$$\mathrm{tr}_n^{Q+\theta}(A_0 \otimes \cdots \otimes A_n) := \mathrm{tr}(\mathcal{R}_n^{\zeta}(A_0 \otimes A_1 \otimes \cdots \otimes A_n)(z))\,|_{z=0}^{\mathrm{mer}}$$

the *quantised Q-weighted trace* of $A_0 \otimes \cdots \otimes A_n \in \mathcal{C}_n(M, E)$.[7]

The above theorem can be reformulated as follows.

Corollary 7. *Given an elliptic operator $Q \in C\ell(M, E)$ with scalar leading symbol, for any positive integer n the quantised Q-weighted trace of $A_0 \otimes \cdots \otimes A_n \in C\ell_n(M, E)$ is local as a finite linear combination of noncommutative residues:*

$$\mathrm{tr}_n^{Q+\theta}(A_0 \otimes \cdots \otimes A_n) = \sum_{0\leq |k|\leq |a|-nq+d} \frac{(-1)^{|k|+n}\, c(k)}{q(|k|+n)} \mathrm{res}(A_0\, A_1^{(k_1)} \cdots A_n^{(k_n)}\, Q^{-(|k|+n)}). \tag{10}$$

For $n = 0$ we have

$$\mathrm{tr}_0^{Q}(A) = \mathrm{tr}^{Q}(A)$$

which in general is non local.

Remark 10. Applying this to the case of a spin manifold with E the spinor bundle and $Q = D^2$ the Laplacian of the associated Dirac operator D we get a local formula similar to that obtained for the Chern character by Connes and

[7] It differs from the weighted trace cochains defined in [24].

Moscovici (see Theorem II.1 of [7]) which is not surprising in view of Remark 7 which relates the Chern character in cyclic cohomology with quantised weighted traces:

$$\begin{aligned}&\mathrm{tr}_n^{Q+\theta}(A_0 \otimes [D, A_1] \otimes \cdots \otimes [D, A_n]) \\ &\quad = \sum_{0\le |k| \le |a|-nq+d} \frac{(-1)^{|k|+n}\, c(k)}{2(|k|+n)} \operatorname{res}(A_0\, [D, A_1]^{(k_1)} \cdots [D, A_n]^{(k_n)}\, Q^{-(|k|+n)}).\end{aligned}$$

3.2. Quantised heat-kernel regularisation

Let us now assume that Q is elliptic with positive leading symbol so that $\mathrm{e}^{-\varepsilon Q}$ defines a smoothing operator for any $\varepsilon > 0$. The heat-kernel regularisation procedure:

$$\mathcal{R}^{Q,HK}(A)(\varepsilon) := A\mathrm{e}^{-\varepsilon Q}, \quad \varepsilon > 0$$

can be quantized to $\mathcal{R}_\bullet^{Q+\theta,HK}$ on $\mathcal{C}_\bullet(M, E)$ defined by

$$\mathcal{R}_n^{Q+\theta,HK}(A_0 \otimes \cdots \otimes A_n)(\varepsilon) := A_0\, f_n(\varepsilon)(Q)(A_1 \otimes \cdots \otimes A_n), \quad \varepsilon > 0$$

with $f_\bullet(\varepsilon)(Q+\theta)$ the quantised version of $f(\varepsilon)(Q) := \mathrm{e}^{-\varepsilon\, Q}$.

Proposition 6. *Let $Q \in C\ell(M, E)$ be an elliptic operator with scalar leading symbol. For any $\varepsilon > 0$ the expression $A_0\,\mathrm{e}^{-\varepsilon(Q+\theta)}(A_1, \ldots, A_n)$ is smoothing for any $A_0, \ldots, A_n \in C\ell(M, E)$ and for any positive integer n*

$$\begin{aligned}\mathcal{R}_n^{Q+\theta,HK}(A_0 \otimes \cdots \otimes A_n)(\varepsilon) &= (-\varepsilon)^n \int_{\Delta_n} A_0\, \mathrm{e}^{-u_0 \varepsilon Q} \cdots A_n\, \mathrm{e}^{-\varepsilon u_n Q} \mathrm{d}u \\ &\sim \sum_{|k|\ge 0} \frac{(-1)^{|k|+n} c(k)\, \varepsilon^{|k|+n}}{(|k|+n)!} A_0\, A_1^{(k_1)} \cdots A_n^{(k_n)} \mathrm{e}^{-\varepsilon Q}.\end{aligned}$$

Proof. The fact that $A_0\,\mathrm{e}^{-\varepsilon(Q+\theta)}(A_1, \ldots, A_n)$ is smoothing together with the first identity follow from the first identity in Corollary 2. The second expression follows from the second identity in Corollary 2. $\square$

The following lemma, which we quote from [12] (see also Lemma 9.34 in [3], formula (3.19) in [18] (here $k = 0$) and formula (1.2) in [1]) is useful to compare heat-kernel and zeta regularisation.

Lemma 3. *Let $f \in C^\infty(]0, \infty[)$ with asymptotic expansion for small t of the form:*

$$f(t) \sim_0 \sum_{j=0}^{\infty} a_j t^{\frac{j-\alpha}{q}} + \sum_{j=0,\, \frac{j-\alpha}{q} \in \mathbb{Z}}^{\infty} b_j t^{\frac{j-\alpha}{q}} \log t + \sum_{j=0}^{\infty} c_j t^j \tag{11}$$

for some real numbers α, a_j, b_j, c_j, $j \in \mathbb{N} \cup \{0\}$ and a positive real number q. Let us moreover assume that it decays exponentially at infinity. Then the Mellin transform:

$$\mathcal{M}(f)(z) := \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} f(t)\mathrm{d}t$$

is a meromorphic function with simple pole at $z = 0$ and whenever α is a non negative integer we have

$$\mathcal{M}(f)(z) = -\frac{b_\alpha}{z} + \mathrm{fp}_{t=0} f(t) - b_\alpha \gamma + o(1)$$

where γ is the Euler constant and $\mathrm{fp}_{t=0} f(t)$ the constant term in the asymptotic expansion (11). In all other cases, the map $z \mapsto \mathcal{M}(f)(z)$ is holomorphic at $z = 0$.

In general we have

$$\mathrm{fp}_{z=0} \mathcal{M}(f)(z) = \mathrm{fp}_{t=0} f(t) - \gamma\, \mathrm{Res}_{z=0} \mathcal{M}(f)(z). \tag{12}$$

Although ordinary heat-kernel regularised traces differ from zeta-regularised traces by a term proportional to the noncommutative residue, the following theorem shows that higher heat-kernel quantised regularised traces coincide with quantised zeta regularised traces.

Theorem 3. *Let $Q \in C\ell(M, E)$ be an elliptic operator with positive order and positive leading symbol. Then, for any $A \in C\ell(M, E)$*

$$\mathrm{tr}^{Q}(A) = \mathrm{fp}_{\varepsilon=0}\mathrm{tr}(\mathcal{R}^{\mathbf{Q},HK}(A)(\varepsilon)) - \frac{\gamma}{q}\,\mathrm{res}(A), \tag{13}$$

so that ordinary heat-kernel and zeta regularised traces coincide on operators with vanishing residue. In particular, they coincide on differential operators.

For any non negative integer n, the map $\varepsilon \mapsto \mathrm{tr}(\mathcal{R}_n^{Q,HK}(A_0 \otimes \cdots \otimes A_n)(\varepsilon))$ has an asymptotic Laurent expansion around 0 in fractional powers of ε.

Its finite part at $\varepsilon = 0$ coincides with the quantised Q-weighted trace of $A_0 \otimes \cdots \otimes A_n$ when the A_i's and Q are differential operators or for any large enough integer n, i.e. such that $|a| + d < |k| + q\,n$ with $|a|$ the total order of the product $A_0 \cdots A_n$, d the dimension of M. In those cases we have:

$$\begin{aligned}\mathrm{fp}_{\varepsilon\to 0}\mathrm{tr}(\mathcal{R}_n^{Q,HK}(A_0 \otimes \cdots \otimes A_n)(\varepsilon)) &= \mathrm{fp}_{\varepsilon\to 0}\mathrm{tr}(A_0\mathrm{e}^{-\varepsilon(Q+\theta)}(A_1 \otimes \cdots \otimes A_n))\\ &= \mathrm{tr}_n^{Q+\theta}(A_0 \otimes \cdots \otimes A_n).\end{aligned}$$

Proof. The first part of the theorem follows from (12) applied to $f(t) = \mathrm{tr}(A\mathrm{e}^{-t\,Q})$ which (see [12], see also [18]) satisfies the assumptions of Lemma 3 with $\alpha = a + d$, where a is the order of A and q the order of Q. Lemma 2 then gives the expression of the complex residue $\mathrm{Res}_{z=0}\mathrm{tr}(A\,Q^{-z})\,|^{\mathrm{mer}}$ in terms of the noncommutative residue.

The second part of the statement follows from Lemma 3 applied to

$$f_k(t) := t^{n+|k|}\,\mathrm{tr}(A_0\,A_1^{(k_1)} \cdots A_n^{(k_n)}\,\mathrm{e}^{-t\,Q})$$

which also satisfies the assumptions of the lemma.

If the A_i's and Q are differential operators, then the pole at 0 of its Mellin transform $\mathcal{M}(f)(z)$ vanishes since the noncommutative residue vanishes on differential operators. The statement of the theorem then follows from (12) combined with Proposition 6.

As for the general case, since Q has scalar leading symbol, the operator $A = A_0\,A_1^{(k_1)} \cdots A_n^{(k_n)}$ has order $|a| + |k|(q-1)$ and the constant α in (11) is $\alpha = |a| + |k|(q-1) + d$. Due to the presence of the multiplicative factor $\varepsilon^{|k|+n}$ in Proposition 6, it follows that $\mathrm{tr}(\mathcal{R}_n^{Q,HK}(A_0 \otimes \cdots \otimes A_n)(\varepsilon))$ has the same type of asymptotic behaviour with α replaced by $|a| + |k|(q-1) + d - q(n + |k|) = |a| - |k| + d - q\,n$ so that there is no logarithmic divergent term and hence no complex residue for large n i.e. when $|a| + d < |k| + q\,n$. The result then follows from (12) combined with Proposition 6. □

4. Anomalies for quantised regularised traces

Regularisation procedures give rise to anomalies due to the presence of the weight Q. Roughly speaking, we show that the anomaly of a quantised regularised trace of level n is a finite linear combination of quantised regularised traces of level $n + 1$. Since quantised regularised traces of any positive level n are local as finite linear combinations of noncommutative residues, so is the anomaly of any quantised regularised trace including those of level 0. We investigate different types of anomalies, the behaviour under the adjoint action of $C\ell(M, E)$, the Hochschild coboundary, and the variation when the weight Q varies.

4.1. Behaviour under the adjoint action

Let $Q \in E\ell\ell^{\mathrm{adm}}_{\mathrm{ord}>0}(M, E)$ and $C \in C\ell(M, E)$. The adjoint action ad_C on $C\ell(M, E)$ induces a transformation of the corresponding ζ-regularisation procedure, namely

$$\mathrm{ad}_C(\mathcal{R}_\bullet^{Q+\theta,\zeta})(z) := \mathrm{ad}_C\left(\theta\,(Q + \pi_Q + \theta)^{-z}\right).$$

The following result shows how the adjoint action of $C\ell(M, E)$ sends quantised ζ-regularised traces of level n to quantised regulalarised ζ-traces of level $n + 1$.

Theorem 4. *Let $Q \in E\ell\ell^{\mathrm{adm}}_{\mathrm{ord}>0}(M,E)$ have scalar leading symbol. Then, for any $C \in C\ell(M,E)$ and any $A_0,\dots,A_n \in C\ell(M,E)$, the operator $\mathrm{ad}_C(\mathcal{R}_\bullet^{Q+\theta,\zeta})(z)(A_0\otimes\cdots\otimes A_n)$ is trace-class for any complex number z with real part large enough and the holomorphic map $z\mapsto \mathrm{tr}(\mathrm{ad}_C(\mathcal{R}_\bullet^{Q+\theta,\zeta})(z)(A_0\otimes\cdots\otimes A_n))$ defined on the corresponding half plane extends to a meromorphic map*

$$z\mapsto \mathrm{tr}(\mathrm{ad}_C(\mathcal{R}_\bullet^{Q+\theta,\zeta})(z)(A_0\otimes\cdots\otimes A_n))\,|^{\mathrm{mer}}$$

defined on the whole complex plane. We denote the value at $z=0$ of the meromorphic extension by:

$$(\mathrm{ad}_C\mathrm{tr}_\bullet^{Q+\theta})(A_0\otimes\cdots\otimes A_n) := \mathrm{tr}(\mathrm{ad}_C(\mathcal{R}_\bullet^{Q+\theta,\zeta})(z)(A_0\otimes\cdots\otimes A_n))\,|^{\mathrm{mer}}_{z=0}.$$

Furthermore,

$$(\mathrm{ad}_C\mathrm{tr}_\bullet^{Q+\theta})(A_0\otimes\cdots\otimes A_n) = \sum_{j=0}^n \mathrm{tr}^{Q+\theta}(A_0\otimes\cdots\otimes A_j\otimes \mathrm{ad}_C(Q)\otimes A_{j+1}\otimes\cdots\otimes A_n).$$

Proof. Applying Corollary 3 to $f(\lambda)=\lambda^{-z}$, by definition of $\mathcal{R}_\bullet^{Q+\theta,\zeta}$ we get

$$\begin{aligned}\mathrm{ad}_C(\mathcal{R}_\bullet^{Q+\theta,\zeta})(z)(A_0\otimes\cdots\otimes A_n) &= \mathrm{ad}_C(\theta\,(Q+\pi_Q+\theta)^{-z})(A_0\otimes\cdots\otimes A_n)\\ &= \sum_{j=0}^n \theta\,(Q+\pi_Q+\theta)^{-z}(A_0\otimes\cdots\otimes A_j\otimes\mathrm{ad}_C(Q)\otimes A_{j+1}\otimes\cdots\otimes A_n)\\ &= \sum_{j=0}^n \mathcal{R}_{n+1}^{Q+\theta,\zeta}(z)(A_0\otimes\cdots\otimes A_j\otimes\mathrm{ad}_C(Q)\otimes A_{j+1}\otimes\cdots\otimes A_n).\end{aligned}$$

By Theorem 2, for any complex number z with real part of z large enough, each of the operators $\mathcal{R}_{n+1}^{Q+\theta,\zeta}(z)(A_0\otimes\cdots\otimes A_j\otimes\mathrm{ad}_C(Q)\otimes A_{j+1}\otimes\cdots\otimes A_n)$ is trace class and each of the maps $z\mapsto \mathrm{tr}(\mathcal{R}_{n+1}^{Q+\theta,\zeta}(z)(A_0\otimes\cdots\otimes A_j\otimes \mathrm{ad}_C(Q)\otimes A_{j+1}\otimes\cdots\otimes A_n))$ extends to a meromorphic map. As a result there is a meromorphic extension

$$z\mapsto \mathrm{tr}(\mathrm{ad}_C(\mathcal{R}_\bullet^{Q+\theta,\zeta})(z)(A_0\otimes\cdots\otimes A_n))\,|^{\mathrm{mer}}$$

defined on the whole plane, the finite part of which we denote by $(\mathrm{ad}_C\mathrm{tr}_\bullet^{Q+\theta})(A_0\otimes\cdots\otimes A_n)$. The statement of the theorem then follows. □

The locality of quantised weighted traces of any positive level n then provides the locality of the transformed regularised traces under the adjoint action.

Corollary 8. *For any $A_0,\dots,A_n \in C\ell(M,E)$, for any $Q\in E\ell\ell^{\mathrm{adm}}_{\mathrm{ord}>0}(M,E)$ with scalar leading symbol and for any $C\in C\ell(M,E)$, the transformed trace under the adjoint action $(\mathrm{ad}_C\mathrm{tr}_n^{Q+\theta})(A_0\otimes\cdots\otimes A_n)$ is local as a linear combination of noncommutative residues:*

$$\begin{aligned}(\mathrm{ad}_C\mathrm{tr}_n^{Q+\theta})(A_0\otimes\cdots\otimes A_n) &= \sum_{j=0}^n\sum_{0\le|k|\le|a|-nq-q+d}\frac{(-1)^{|k|+n+1}\,c(k)}{q(|k|+n+1)}\\ &\quad\times \mathrm{res}(A_0\,A_1^{(k_1)}\cdots A_j^{(k_j)}C^{(k_{j+1})}\,A_{j+1}^{(k_{j+2})}\cdots A_n^{(k_n)}\,Q^{-(|k|+n+1)}).\end{aligned}$$

Proof. The result follows from Theorem 4 combined with formula (10). □

4.2. Hochschild coboundaries

The coboundary of an ordinary ζ-regularised trace $\delta(\mathrm{tr}^Q)(A,B) := \mathrm{tr}^Q([A,B])$ generalises to the Hochschild coboundary for higher level quantised ζ-regularised trace cochains:

$$\begin{aligned}(b\mathrm{tr}_n^{Q+\theta})(A_0\otimes\cdots\otimes A_{n+1}) &:= \sum_{j=0}(-1)^j\mathrm{tr}_n^{Q+\theta}(A_0\otimes\cdots\otimes A_jA_{j+1}\otimes\cdots\otimes A_{n+1})\\ &\quad + (-1)^{n+1}\mathrm{tr}_n^{Q+\theta}(A_{n+1}A_0\otimes\cdots\otimes A_j\otimes\cdots\otimes A_n). \qquad (14)\end{aligned}$$

The following result shows that the Hochschild coboundary of a quantised ζ-regularised trace is a finite linear combination of quantised ζ-regularised traces of the same level or higher.

Proposition 7. *Let $Q \in Ell^{\mathrm{adm}}_{\mathrm{ord}>0}(M,E)$ have scalar leading symbol. For any non negative integer p and any $A_0,\ldots,A_{2p+1} \in C\ell(M,E)$, on the one hand,*

$$(b\,\mathrm{tr}^{Q+\theta}_{2p})\,(A_0\otimes\cdots\otimes A_{2p+1}) = \sum_{j=0}^{p} \mathrm{tr}^{Q+\theta}_{2p+1}\,(A_0\otimes\cdots\otimes A_{2j}\otimes[Q,A_{2j+1}]\otimes A_{2j+2}\otimes\cdots\otimes A_{2p+1}).$$

On the other hand,

$$\begin{aligned}(b\,\mathrm{tr}^{Q+\theta}_{2p+1})\left(A_0\otimes\cdots\otimes A_{2p+2}\right) &= \sum_{j=0}^{p} \mathrm{tr}^{Q+\theta}_{2p+2}\,(A_0\otimes\cdots\otimes A_{2j}\otimes[Q,A_{2j+1}]\otimes A_{2j+2}\otimes\cdots\otimes A_{2p+2})\\ &\quad + \mathrm{tr}^{Q+\theta}_{2p+1}\,(A_{2p+2}\,A_0\otimes\cdots\otimes A_{2p+1}).\end{aligned}$$

Proof. Following [14] (see e.g. Lemma 5.2), we first check that

$$\begin{aligned}&R_n(\lambda,Q+\theta)\,(A_0\otimes\cdots\otimes A_{j-1}A_j\otimes\cdots\otimes A_{n+1}) - R_n(\lambda,Q+\theta)\,(A_0\otimes\cdots\otimes A_jA_{j+1}\otimes\cdots\otimes A_{n+1})\\ &\quad = R_{n+1}(\lambda,Q+\theta)\,(A_0\otimes\cdots\otimes[Q,A_j]\otimes\cdots\otimes A_{n+1}),\end{aligned}$$

which easily follows from the identity

$$A_{j-1}(\lambda-Q)^{-1}[Q,A_j](\lambda-Q)^{-1}A_{j+1} = A_{j-1}(\lambda-Q)^{-1}A_jA_{j+1} - A_{j-1}A_j(\lambda-Q)^{-1}A_{j+1}.$$

The result then follows using (14). $\square$

As a consequence, the Hochschild coboundary of a quantised ζ-regularised trace is local.

Theorem 5. *Let $Q \in Ell^{\mathrm{adm}}_{\mathrm{ord}>0}(M,E)$ have scalar leading symbol. For any non negative integer p and any $A_1,\ldots,A_{2p+2} \in C\ell(M,E)$, the Hochschild coboundary of a ζ-regularised trace is a finite linear combination of noncommutative residues.*

$$\begin{aligned}(b\,\mathrm{tr}^{Q+\theta}_{2p})(A_0\otimes\cdots\otimes A_{2p+1}) &= \sum_{j=0}^{p}\sum_{0\le|k|\le|a|-2p\,q-q+d} (-1)^{|k|+1}\frac{c(k)}{q(|k|+2p+1)}\\ &\quad\times \mathrm{res}(A_0\,A_1^{(k_1)}\cdots A_{2j+1}^{(k_{2j+1}+1)}\cdots A_{2p+1}^{(k_{2p+1})}\,Q^{-(|k|+2p+1)}),\end{aligned}$$

and

$$\begin{aligned}(b\,\mathrm{tr}^{Q}_{2p+1})\left(A_0\otimes\cdots\otimes A_{2p+2}\right) &= \sum_{j=1}^{p}\sum_{0\le|k|\le|a|-2pq-2q+d} (-1)^{|k|}\frac{c(k)}{q(|k|+2p+2)}\\ &\quad\times \mathrm{res}(A_0\,A_1^{(k_1)}\cdots A_{2j+1}^{(k_{2j+1}+1)}\cdots A_{2p+2}^{(k_{2p+2})}\,Q^{-(|k|+2p+2)})\\ &\quad+\sum_{0\le|k|\le|a|-2pq-q+d} (-1)^{|k|+1}\frac{c(k)}{q(|k|+2p+1)}\\ &\quad\times \mathrm{res}(A_{2p}A_0A_1^{(k_1)}\cdots A_{2p-1}^{(k_{2p-1})}\,Q^{-(|k|+2p+1)}).\end{aligned}$$

Proof. This follows from Proposition 7 combined with formula (10). The locality then follows from the locality of the noncommutative residue. $\square$

Remark 11. When $p=0$ we get that

$$b\mathrm{tr}^{Q}(A\otimes B) = \sum_{0\le k\le a-q+d} \frac{(-1)^{k+1}}{q(k+2)}\,\mathrm{res}(AB^{(k+1)}\,Q^{-(k+2)}).$$

4.3. Variation of the weight

A third type of anomaly arises from letting the weight vary. Let $Q(b), b \in B$ be a differentiable family in $E\ell\ell^{\mathrm{adm}}_{\mathrm{ord}>0}(M, E)$ of fixed order $q > 0$ parametrised by a manifold B.

Theorem 6. *Let $Q(b), b \in B$ be a differentiable family parametrised by a manifold B in $E\ell\ell^{\mathrm{adm}}_{\mathrm{ord}>0}(M, E)$ with scalar leading symbol and fixed order $q > 0$. Then, for any fixed operators $A_0, \ldots, A_n \in C\ell(M, E)$, the operator $d(\mathcal{R}^{Q+\theta,\zeta}_{\bullet})(z)(A_0 \otimes \cdots \otimes A_n)$ is trace-class for real part of z large enough and the holomorphic map $z \mapsto \mathrm{tr}(d(\mathcal{R}^{Q+\theta,\zeta}_{\bullet})(z)(A_0 \otimes \cdots \otimes A_n))$ defined on the corresponding half plane extends to a meromorphic map*

$$z \mapsto \mathrm{tr}(d(\mathcal{R}^{Q+\theta,\zeta}_{\bullet})(z)(A_0 \otimes \cdots \otimes A_n))\,|^{\mathrm{mer}}$$

defined on the whole complex plane. This meromorphic extension turns out to be holomorphic at $z = 0$ and we denote its value at 0 by:

$$(\mathrm{dtr}^{Q+\theta}_{\bullet})(A_0 \otimes \cdots \otimes A_n) := \mathrm{tr}(d(\mathcal{R}^{Q+\theta,\zeta}_{\bullet})(z)(A_0 \otimes \cdots \otimes A_n))\,|^{\mathrm{mer}}_{z=0}.$$

Furthermore, exterior differentials of quantised ζ-regularised traces of level n are finite linear combinations of quantised ζ-regularised traces of level $n+1$. For any non negative integer n:

$$(\mathrm{dtr}^{Q+\theta}_{n})(A_0 \otimes \cdots \otimes A_n) = \sum_{j=0}^{n} \mathrm{tr}^{Q+\theta}_{n+1}\,(A_0 \otimes \cdots \otimes A_j \otimes \mathrm{d}Q \otimes A_{j+1} \otimes \cdots \otimes A_n)$$

for any $A_0, \ldots, A_n \in C\ell(M, E)$.

Proof. This follows from Corollary 5 applied to $f(\lambda) = \lambda^{-z}$ combined with Theorem 2. □

As a result, the exterior differentials of quantised ζ-regularised traces are local.

Corollary 9. *Let $Q(b), b \in B$ be a differentiable family in $E\ell\ell^{\mathrm{adm}}_{\mathrm{ord}>0}(M, E)$ of fixed order $q > 0$ parametrised by a manifold B. Exterior differentials of quantised ζ-regularised traces of level n are local*

$$\begin{aligned}\mathrm{dtr}^{Q}_{n}(A_0 \otimes \cdots \otimes A_n) = &\sum_{j=0}^{n} \sum_{0\le |k| \le |a|-nq-q+d} \frac{c(k)}{q(|k|+n+1)}(-1)^{|k|+n+1} \\ &\times \mathrm{res}(A_0\, A_1^{(k_1)} \cdots A_j^{(k_j)}\, (\mathrm{d}Q)^{(k_{j+1})} A_{j+1}^{(k_{j+2})} \cdots A_n^{(k_n)}\, Q^{-(|k|+n+1)})\end{aligned}$$

for any $A_0 \otimes \cdots \otimes A_n \in \mathcal{C}_{\bullet}(M, E)$.

Proof. This follows from Theorem 6 combined with formula (10). □

5. The family index theorem setup

We provide the basic ingredients that enable a generalisation of quantised ζ-regularised traces to a family index theorem setup and will only sketch the actual extension, referring to [26] for further details. We adopt the notations of [26] from which we quote some of the preliminary results.

Consider a smooth fibration $\pi : M \to B$ with closed finite dimensional fibre $M_b := \pi^{-1}(b)$ equipped with a Riemannian metric $g_{M/B}$ on the tangent bundle $T(M/B)$. Let $|\Lambda_\pi| = |\Lambda(T^*(M/B))|$ be the line bundle of vertical densities, restricting on each fibre to the usual bundle of densities $|\Lambda_{M_b}|$ along M_b. Let $\mathcal{E} := \mathcal{E}^+ \oplus \mathcal{E}^-$ be a vertical Hermitian $\mathbb{Z}_2$-graded vector bundle over M and let $\pi_*(\mathcal{E}) := \pi_*(\mathcal{E}^+) \oplus \pi_*(\mathcal{E}^-)$ be the graded infinite dimensional Fréchet bundle with fibre $C^\infty(M_b, \mathcal{E}_b \otimes |\Lambda_{M_b}|^{\frac{1}{2}})$ at $b \in B$, where $\mathcal{E}_b$ is the $\mathbb{Z}_2$-graded vector bundle over M_b obtained by restriction of $\mathcal{E}$. By definition, a smooth section σ of $\pi_*(\mathcal{E})$ over B is a smooth section of $\mathcal{E} \otimes |\Lambda_\pi|^{\frac{1}{2}}$ over M, so that $\sigma(b) \in C^\infty(M_b, \mathcal{E}_b \otimes |\Lambda_{M_b}|^{\frac{1}{2}})$ for all $b \in B$. More generally, the de Rham complex of smooth forms on B with values in $\pi_*(\mathcal{E})$ is defined by:

$$\mathcal{A}(B, \pi_*(\mathcal{E})) = C^\infty\left(M, \pi^*(\wedge T^*B) \otimes \mathcal{E} \otimes |\Lambda_\pi|^{\frac{1}{2}}\right)$$

with $\otimes$ the $\mathbb{Z}_2$-graded tensor product. Let $C\ell(\mathcal{E})$ denote the infinite-dimensional bundle of algebras with fibre $C\ell(\mathcal{E}_b) = C\ell(M_b, \mathcal{E}_b \otimes |\Lambda_{M_b}|^{\frac{1}{2}})$. A section $\mathbf{Q} \in \mathcal{A}(B, C\ell(\mathcal{E}))$ defines a smooth family of classical pseudodifferential operators with differential form coefficients parametrised by B. Such an operator valued form $\mathbf{Q}$ is locally described by a *vertical* symbol

$$\mathbf{q}(x, y, \xi) \in C^{\infty}\left((U_M \times_{\pi} U_M) \times \mathbb{R}^n, \pi^*(\Lambda T^* U_B) \otimes \mathbb{R}^N \otimes \left(\mathbb{R}^N\right)^*\right),$$

where $\times_\pi$ is the fibre product, ξ may be identified with a vertical vector in $T_b(M/B)$, and U_M is a local coordinate neighbourhood of M over which $\mathcal{E}_{U_M} \simeq U_M \times \mathbb{R}^N$ is trivialised and $\mathbb{R}^N$ inherits the grading of $\mathcal{E}$. With respect to the local trivialisation of $\pi_*(\mathcal{E})$ over $U_B = \pi(U_M)$ one has

$$\mathcal{A}\left(U, \pi_*(\mathcal{E})_{|U_B}\right) \simeq \mathcal{A}(U) \otimes C^{\infty}\left(M_{b_0}, \mathcal{E}^{b_0}\right)$$

with $M_{b_0} = \pi^{-1}(b_0)$ relative to a base point $b_0 \in U_B$, so that $\mathbf{q}$ can be written locally over U_B as a finite sum of terms of the form $\omega_k \otimes \mathbf{q}_{[k]}$, where $\omega_k \in \mathcal{A}^k(U_B)$ and $\mathbf{q}_{[k]} \in C^{\infty}(U_{b_0} \times \mathbb{R}^n/\{0\}, \mathbb{R}^N \times (\mathbb{R}^N)^*)$ is a symbol (in the single manifold sense) of form degree zero. We will work only with local symbols which have the local form $\sum_{k=0}^{\mathrm{dimB}} \omega_k \otimes \mathbf{q}_{[k]}$, with just one term in each form degree, extending by linearity to general sums. The order of a such a symbol is defined to be the $(\dim B + 1)$-tuple $(q_0, \ldots, q_{\dim B})$ with q_k the order of the symbol $\mathbf{q}_{[k]}$; for simplicity we consider the case where *q_k is constant on B*. In accordance with the splitting of the local symbol into form degree $\mathbf{q} = \mathbf{q}_{[0]} + \cdots + \mathbf{q}_{[\dim B]}$, the operator

$$(\mathbf{Q}\psi)(x) = \frac{1}{(2\pi)^n} \int_{M/B} \mathrm{d}\,\mathrm{vol}_{M/B} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}(x-y)\cdot\xi} \mathbf{q}(x, y, \xi) \psi(y) \mathrm{d}\xi,$$

for ψ with compact support in U_M, splits as $\mathbf{Q} = \mathbf{Q}_{[0]} + \mathbf{Q}_{[1]} + \cdots + \mathbf{Q}_{[\dim B]}$, where $\mathbf{Q}_{[k]} = \omega_k \otimes Q_k \in \mathcal{A}(U_B, \pi_*(\mathcal{E})_{|U_B})$ in each form degree.

Definition 8. A smooth family $\mathbf{Q} \in \mathcal{A}(B, C\ell(\mathcal{E}))$ of vertical pseudodifferential operators is elliptic if its form degree zero component $\mathbf{Q}_{[0]}$ is pointwise (with respect to the parameter manifold B) elliptic.

In this case $\mathbf{Q}$ has spectral cut θ if $\mathbf{Q}_{[0]}$ admits a spectral cut θ. Likewise, it is invertible if $\mathbf{Q}_{[0]}$ is invertible. Setting $\mathbf{Q}_{[>0]} := \mathbf{Q} - \mathbf{Q}_{[0]} \in \mathcal{A}^1(B, C\ell(\mathcal{E}))$, for λ outside the spectrum of $\mathbf{Q}_{[0]}$

$$\begin{aligned} R(\lambda, \mathbf{Q}) &= (\lambda - \mathbf{Q})^{-1} \\ &= \left(\lambda - (\mathbf{Q}_{[0]} + \mathbf{Q}_{[>0]})\right)^{-1} \\ &= \sum_{k=1}^{\dim B} R_k(\lambda, \mathbf{Q} + \Theta)(\mathbf{Q}_{[>0]}, \ldots, \mathbf{Q}_{[>0]}) \quad \text{with } \mathbf{Q}_{[>0]} \text{ arising } k \text{ times.} \end{aligned} \tag{15}$$

In particular $((\mathbf{Q} - \lambda)^{-1})_{[0]} = (\mathbf{Q}_{[0]} - \lambda)^{-1}$.

Without giving a detailed description of the analog in this family setup of the constructions carried out in the ordinary setup, let us however give the basic ingredients that lead to a generalisation of the quantised ζ-regularised traces to the family setup, using $\mathbf{Q}$ as a weight.

Let $\mathbf{Q}$ be a smooth family of vertical admissible elliptic invertible ΨDOs, the orders $(q_0, \ldots, q_{\dim B+1})$ of which fulfill the assumption

$$q_0 = \mathrm{ord}(Q_{[0]}) > 0$$

and

$$q_k \le q_0 \quad \forall k \ge 1. \tag{16}$$

Under these assumptions one obtains an operator norm estimate in $\mathcal{A}(B)$ as $|\lambda| \to \infty$ in Γ_θ

$$\|(\lambda I - \mathbf{Q})^{-1}\|_{M/B}^{(s)} = O(|\lambda|^{-1}) \quad \forall s \in \mathbb{R}$$

where $\|\cdot\|_{M/B}^{(l)} : \mathcal{A}(B, C\ell(\mathcal{E})) \to \mathcal{A}(B)$ is the vertical Sobolev endomorphism norm associated to the vertical metric.

From there, we can mimic the construction of the quantised resolvent for ordinary pseudodifferential operators and define the quantised resolvent $R(\lambda, \mathbf{Q}) = (\lambda - (\mathbf{Q} + \Theta))^{-1}$ of $\mathbf{Q}$ as well as its quantised complex power $(\mathbf{Q} + \Theta)^{-z}$ when $\mathbf{Q}$ has a spectral cut where Θ is now the insertion operator $\Theta(\alpha) = \alpha$ for all $\alpha \in \mathcal{A}(B, \pi_*(\mathcal{E}))$.

As in [26], where ordinary weighted traces were extended to pseudodifferential operator valued forms $\alpha \in \mathcal{A}(B, \pi_*(\mathcal{E}))$, quantised weighted traces and noncommutative residues can be extended to pseudodifferential valued forms $\alpha_i \in \mathcal{A}(B, \pi_*(\mathcal{E}))$. Theorem 2 applied fibrewise above each fibre b then yields the following statements:

Proposition 8. *Let $\mathbf{Q} \in \mathcal{A}(B, C\ell(\mathcal{E}))$ be a vertical elliptic operator valued even form with scalar leading symbol and positive order q. For any $\alpha_1, \ldots, \alpha_n \in \mathcal{A}(B, \pi_*(\mathcal{E}))$, the operator $(\mathbf{Q} + \pi_{\mathbf{Q}_{[0]}} + \Theta)^{-z}(\alpha_1 \otimes \cdots \otimes \alpha_n)$ is trace-class for real part of z large enough and the map $z \mapsto \operatorname{tr}(\Theta\,(\mathbf{Q} + \pi_{\mathbf{Q}_{[0]}} + \Theta)^{-z}(\alpha_0 \otimes \cdots \otimes \alpha_n))$ extends to a meromorphic map $z \mapsto \operatorname{tr}(\Theta\,(\mathbf{Q} + \pi_{\mathbf{Q}_{[0]}} + \Theta)^{-z}(\alpha_0 \otimes \cdots \otimes \alpha_n))\,|^{\mathrm{mer}}$ defined on the whole plane. The $\mathbf{Q} + \Theta$-weighted trace of $\alpha_0 \otimes \cdots \otimes \alpha_n$ corresponds to its finite part at 0*

$$\operatorname{tr}_n^{\mathbf{Q}+\Theta}(\alpha_0 \otimes \cdots \otimes \alpha_n) := \operatorname{tr}(\Theta\,(\mathbf{Q} + \pi_{\mathbf{Q}_{[0]}} + \Theta)^{-z}(\alpha_0 \otimes \cdots \otimes \alpha_n))\,|^{\mathrm{mer}}_{z=0}.$$

For any positive integer n, the quantised ζ-trace $\operatorname{tr}_n^{\mathbf{Q}+\Theta}$ is local as a linear combination of noncommutative residues:

$$\operatorname{tr}_n^{\mathbf{Q}+\Theta}(\alpha_0 \otimes \cdots \otimes \alpha_n) = \sum_{0 \le |k| \le |a| - nq + d} \frac{(-1)^{|k|+n} c(k)}{q(|k|+n)} \times \operatorname{res}\left(\alpha_0 \wedge \alpha_1^{(k_1)} \wedge \cdots \wedge \alpha_n^{(k_n)} \wedge (\mathbf{Q} + \pi_{\mathbf{Q}_{[0]}})^{-(|k|+n)}\right).$$

Remark 12. With these notations and those of [26] we have $\operatorname{tr}^{\mathbf{Q}}(\alpha) = \operatorname{tr}_0^{\mathbf{Q}}(\alpha)$.

Since we are interested in the variation of quantised weighted traces under a variation of $\mathbf{Q}$, given a connection $\mathbb{A}$ on $\mathcal{A}$ we set for any $\alpha_0, \ldots, \alpha_n \in \mathcal{A}(B, \pi_*(\mathcal{E}))$,

$$\left(d\operatorname{tr}_n^{\mathbf{Q}+\Theta}\right)(\alpha_0 \otimes \cdots \otimes \alpha_n) := d\left(\operatorname{tr}_n^{\mathbf{Q}+\Theta}(\alpha_0 \otimes \cdots \otimes \alpha_n)\right) - \sum_{j=0}^{n} (-1)^{d_0 + \cdots + d_j} \operatorname{tr}_n^{\mathbf{Q}+\Theta}\left(\alpha_0 \otimes \cdots \otimes \alpha_{j-1} \otimes [\mathbb{A}, \alpha_j] \otimes \alpha_{j+1} \otimes \cdots \alpha_n\right). \quad (17)$$

Theorem 7. *Let $\mathbf{Q} \in \mathcal{A}(B, C\ell(\mathcal{E}))$ be a vertical elliptic operator valued even form with scalar leading symbol and positive order q. Let $\mathbb{A}$ be a superconnexion on $\pi_*\mathcal{E}$. For any $\alpha_0, \ldots, \alpha_n \in \mathcal{A}(B, \pi_*(\mathcal{E}))$ of form degrees $d_0, \ldots, d_n$ respectively, we have:*

$$\left(d\operatorname{tr}_n^{\mathbf{Q}}\right)(\alpha_1 \otimes \cdots \otimes \alpha_n) = \sum_{j=0}^{n} (-1)^{|d_j|} \operatorname{tr}_n^{\mathbf{Q}}\left(\alpha_0 \otimes \alpha_1 \otimes \cdots \otimes \alpha_j \otimes [\mathbb{A}, \mathbf{Q}] \otimes \alpha_{j+1} \otimes \cdots \otimes \alpha_n\right),$$

with $|d_j| = d_0 + \cdots + d_j$ as before.

Proof. Locally we write $\mathbb{A} = d + \Phi$ and hence $[\mathbb{A}, \cdot] = d + [\Phi, \cdot]$. Theorem 6 with Theorem 4 (with C replaced by an even degree form) both generalise to pseudodifferential valued forms. Combining them (with C replaced by Φ) yields the result. □

Corollary 10. *Let $\mathbf{Q} \in \mathcal{A}(B, C\ell(\mathcal{E}))$ be a vertical elliptic operator valued even form with scalar leading symbol and positive order q. Let $\mathbb{A}$ be a superconnexion on $\pi_*\mathcal{E}$. For any $\alpha_0, \ldots, \alpha_n \in \mathcal{A}(B, \pi_*(\mathcal{E}))$ of operator orders $a_0, \ldots, a_n$ and form degrees $d_0, \ldots, d_n$ respectively, we have:*

$$\left(d\operatorname{tr}_n^{\mathbf{Q}}\right)(\alpha_1 \otimes \cdots \otimes \alpha_n) = \sum_{j=0}^{n} (-1)^{|d_j|} \sum_{0 \le |k| \le |a| - nq - q + d} \frac{(-1)^{|k|+n+1} c(k)}{q(|k|+n+1)} \times \operatorname{res}\left(\alpha_0 \wedge \alpha_1^{(k_1)} \wedge \cdots \wedge \alpha_{j-1}^{(k_{j-1})} \wedge [\mathbb{A}, \mathbf{Q}]^{(k_j)} \wedge \alpha_j^{(k_{j+1})} \times \wedge \cdots \wedge \alpha_n^{(k_n)} \wedge (\mathbf{Q} + \pi_{\mathbf{Q}_{[0]}})^{-(|k|+n+1)}\right),$$

where $|a| = a_0 \cdots + a_n$ and as before $|d_j| := d_1 + \cdots + d_j$.

Proof. As before, we write $\mathbb{A} = d = \Phi$ and hence $[\mathbb{A}, \cdot] = d + [\Phi, \cdot]$. Combining Corollary 9 with Corollary 8 (with C replaced by Φ) yields the result. $\square$

A superconnection [28,2,3] on $\pi_*\mathcal{E}$ adapted to a smooth family of formally self-adjoint elliptic ΨDOs $Q \in \mathcal{A}^0(B, C\ell^q(\mathcal{E}))$ with odd parity is a classical ΨDO$\mathbb{A}$ on $\mathcal{A}(B, \pi_*\mathcal{E})$ of odd parity with respect to the $\mathbb{Z}_2$-grading such that:

$$\mathbb{A}(\omega \cdot \sigma) = \mathrm{d}\omega \wedge \sigma + (-1)^{|\omega|}\omega \wedge \mathbb{A}(\sigma) \quad \forall \omega \in \mathcal{A}(B), \sigma \in \mathcal{A}(B, \pi_*\mathcal{E})$$

and

$$\mathbb{A}_{[0]} := Q$$

where $\mathbb{A} = \sum_{i=0}^{\dim B} \mathbb{A}_{[i]}$ and $\mathbb{A}_{[i]} : \mathcal{A}^*(B, \pi_*\mathcal{E}) \mapsto \mathcal{A}^{*+i}(B, \pi_*\mathcal{E})$.

Theorem 7 easily extends when replacing **Q** by $\mathbb{A}^2$ so that we get back the expected covariance for ordinary $\mathbb{A}^2$-weighted traces:

Corollary 11. *Let $\mathbb{A}$ be a superconnection on $\pi_*\mathcal{E}$ adapted to a smooth family of formally self-adjoint elliptic ΨDOs $Q \in \mathcal{A}^0(B, C\ell^q(\mathcal{E}))$ then the ordinary $\mathbb{A}^2$ weighted trace is covariantly constant:*

$$\mathrm{dtr}^{\mathbb{A}^2+\Theta} = 0$$

with the notations of (17).

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